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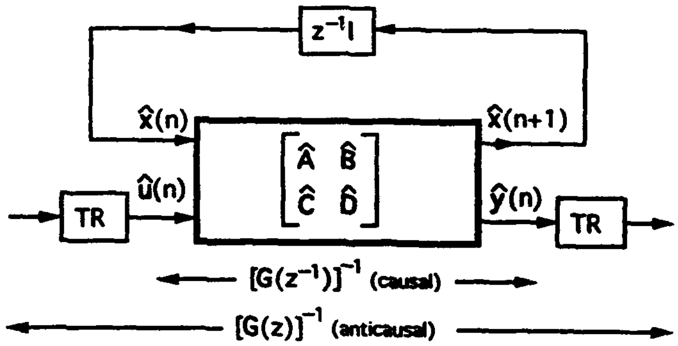
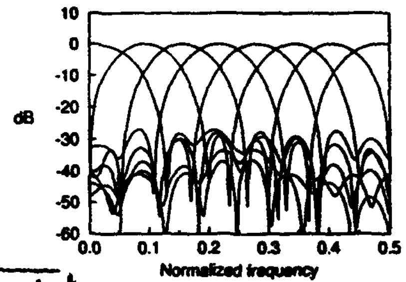


The BOLT filter bank



Part I: System-theoretic fundamentals

Part II: The FIR case, factorizations, and BiOrthonormal Lapped Transforms (BOLT)

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ROLE OF ANTICAUSAL INVERSES IN MULTIRATE FILTER-BANKS — PART I: SYSTEM-THEORETIC FUNDAMENTALS

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Abstract. In a maximally decimated filter bank with identical decimation ratios for all channels, the perfect reconstructibility properties depend on the properties of the polyphase martrix. Various properties and capabilities of the filter bank depend on the properties of the polyphase matrix as well as the nature of its inverse. In this paper we undertake a study of the types of inverses and characterize them according to their system theoretic properties (i.e., properties of state-space descriptions, McMillan degree, degree of determinant, and so forth). We find in particular that causal polyphase matrices with anticausal inverses have an important role in filter bank theory. We study their properties both for the FIR and IIR cases. Techniques for implementing anticausal IIR inverses based on state space descriptions are outlined. It is found that causal FIR matrices with anticausal FIR inverses (abbreviated *cafacafi*) have a key role in the characterization of FIR filter banks. In a companion paper [1] these results are applied for the factorization of biorthonormal FIR filter banks, and a generalization of the lapped orthogonal transform called the biorthonormal lapped transform (*BOLT*) developed.

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1. INTRODUCTION

The M-channel maximally decimated analysis/synthesis system of Fig. 1.1(a) has been studied extensively and used in a number of applications. Extensive references can be found in [2]-[11]. It is known (e.g., see pp. 230–234 of [8]) that this can be redrawn as in Fig. 1.1(b) where $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are the polyphase matrices of the analysis and synthesis bank respectively. This system has the perfect reconstruction (PR) property if, in absence of the subband quantizers Q, we have $\widehat{x}(n) = x(n)$. This is equivalent to the requirement $\mathbf{R}(z)\mathbf{E}(z) = \mathbf{I}$, that is,

$$\mathbf{R}(z) = \mathbf{E}^{-1}(z). \tag{1.1}$$

Thus, as is well-known, the perfect reconstruction problem in an M channel uniform filter bank is related to the invertibility of the polyphase matrix $\mathbf{E}(z)$. Several classes of filter banks have been developed in the past, depending on the nature of $\mathbf{E}(z)$ and its inverse. For example, an FIR filter bank is one where $\mathbf{E}(z)$ and its inverse are FIR. Further examples are:

1. FIR filter banks where the polyphase matrix $\mathbf{E}(z)$ is a cascade of constant nonsingular matrices separated by delays (Fig. 1.2). Here $\mathbf{E}(z)$ is causal and its inverse anticausal [12], [13].
2. FIR paraunitary filter banks [6] where $\mathbf{E}(z)$ is unitary on the unit circle and $\mathbf{R}(e^{j\omega}) = \mathbf{E}^{\dagger}(e^{j\omega})$ (transposed conjugate). When $\mathbf{E}(z)$ is causal, it can be expressed as the cascade in Fig. 1.2 with the matrices $\mathbf{T}_i$ constrained to be unitary. As in the first case, if $\mathbf{E}(z)$ is causal then the inverse is anticausal.
3. FIR filter banks where $\mathbf{E}(z)$ cannot be expressed as the cascade shown in Fig. 1.2. An example is the second linear phase FIR PR system reported in [13], and reproduced in [8] (Prob. 7.3). This is an example where $\mathbf{E}(z)$ is causal but the inverse is neither causal nor anticausal.
4. *IIR paraunitary filter banks.* In the IIR paraunitary case, if $\mathbf{E}(z)$ is causal and stable then its inverse has to be anticausal in order to be stable. For some special cases, techniques to implement such anticausal inverses with finite latency have been discussed before [14]-[18].
5. *Causal IIR filter banks* where the analysis and synthesis filters are both causal and stable, which permit a delay between the input and output [19].

Maximally decimated paraunitary filter banks are also orthonormal filter banks, whereas more generally a maximally decimated PR filter bank is biorthonormal [11]. The first and third examples above are not orthonormal but only biorthonormal. The above results have appeared in the literature as specific instances of PR systems.

There does not appear to be any literature in digital signal processing which gives a general treatment of inverses of transfer matrices, and classifies them according to the type of inverse. In this paper we do this, and derive *system-theoretic* characterizations for various cases. To give an example, we will show that a causal system has an anticausal inverse if and only if the so-called *realization matrix* is nonsingular (Theorem 5.1). We also show that a causal FIR system $\mathbf{E}(z)$ has an anticausal FIR inverse if and only if the degree of the determinant is equal to the McMillan degree, that is $[\det \mathbf{E}(z)] = cz^{-N}$ where N is the McMillan degree of $\mathbf{E}(z)$. Both of these results will have applications: the first one in the stable implementation of an anticausal IIR synthesis bank and the second one in the characterization of a new class of FIR filter banks called the *biorthonormal lapped transform* (*BOLT*).

In our discussions causal transfer matrices with anticausal inverses, especially *CAusal Fir systems with AntiCAusal Fir Inverses* (abbreviated as *cafacafi*) will play a crucial role. This is because essentially any FIR biorthonormal filter bank (with uniform decimation ratios) can be covered with polyphase transfer matrices of this type (Sec. 1.1).

In a companion paper [1] we will present applications of some of these results, for the case of FIR filter banks. For example we will consider the problem of factorizing *cafacafi* systems into degree-one building blocks. In particular a complete characterization of the biorthonormal lapped transform will be presented. We will show examples of *cafacafi* systems which cannot be factorized into degree one systems. We also show that any causal FIR system $\mathbf{E}(z)$ with an FIR inverse can always be factorized as $\mathbf{G}_{c,a}(z)\mathbf{G}_{c,c}(z)$ where $\mathbf{G}_{c,a}(z)$ is causal FIR with anticausal FIR inverse, and $\mathbf{G}_{c,c}(z)$ is causal FIR with causal FIR inverse. However, we will see that while causal FIR systems with anticausal FIR inverses can be factorized under some conditions, unimodular matrices often cannot be factorized into convenient degree one systems.

1.1. Causal LTI systems with anticausal inverses

In this paper we will discuss the theory of causal linear time invariant (LTI) systems with anticausal inverses. It turns out that these systems have an important role in maximally decimated PR (i.e., biorthonormal) filter banks (both FIR and IIR). In fact essentially all FIR PR systems can be characterized with these systems. Furthermore, unlike arbitrary FIR systems with FIR inverses, the system-theoretic properties and factorizability of these systems are more tractable and elegant.

A causal $p \times r$ transfer matrix

$$\mathbf{G}(z) = \sum_{n=0}^{\infty} z^{-n}\mathbf{g}(n) \tag{1.2}$$

is said to have an anticausal inverse if there exists an $r \times p$ transfer matrix

$$\mathbf{H}(z) = \sum_{n=-\infty}^{0} z^{-n}\mathbf{h}(n) = \sum_{n=0}^{\infty} z^{n}\mathbf{h}(-n) \tag{1.3}$$

such that $\mathbf{H}(z)\mathbf{G}(z) = \mathbf{I}_r$. Thus $\mathbf{H}(z)$ is the left inverse of $\mathbf{G}(z)$. We are mostly interested in the $M \times M$ case (i.e., $p = r = M$) because in maximally decimated filter banks, the polyphase matrix $\mathbf{E}(z)$ is $M \times M$.

For the non maximally decimated case, we have $p > r$ with p representing the number of channels and r the decimation ratios. The relevance and importance of causal systems with anticausal inverses depends on whether we are considering the FIR or the IIR case.

Relevance in the Case of IIR Filter Banks

The idea of using an anticausal inverse in order to implement stable IIR filter banks is well-known, and was first proposed in [14] for image subband coding where the inputs have finite length. The fact that IIR inverses can be implemented in an anticausal fashion if their state variables are properly initialized was pointed out in [16]. A general theory of anticausal inversion was then presented in [18]. The (somewhat counterintuitive) result that such anticausal inversion for the IIR case can be performed even with infinitely long input sequences was pointed out in [17]. † In Sec. 3.2 we will review these ideas in the most general setting of a state space formulation.

Relevance in the Case of FIR Filter Banks

In an FIR filter bank, the polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are both FIR. In this case the PR condition (1.1) is satisfied if and only if $\mathbf{E}(z)$ has the property

$$\det \mathbf{E}(z) = cz^{-J} \tag{1.4}$$

for some $c \neq 0$ and integer J. An important problem in this context is the characterization or parameterization of all FIR transfer matrices $\mathbf{E}(z)$ having the above property. There has been some progress in the past [20], and there are many FIR examples in the literature satisfying (1.4) (any orthonormal or biorthonormal FIR PR system is a valid example). However, the general characterization is still an open problem.

Given an FIR PR filter bank with polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ satisfying (1.1), suppose we define a new filter bank with polyphase matrices $\mathbf{E}_1(z) = z^{-I}\mathbf{E}(z)$ and $\mathbf{R}_1(z) = z^{I}\mathbf{R}(z)$ where I is an arbitrary integer. Then the system is still FIR PR, with new FIR analysis filters $z^{-IM}H_k(z)$ and FIR synthesis filters $z^{IM}F_k(z)$. For large enough I, we see that $\mathbf{E}_1(z)$ is causal and its inverse $\mathbf{R}_1(z)$ anticausal. The filter responses are unaffected except for a delay, and this does not affect important properties of the filter bank (e.g., energy compaction, coding gain, etc.). Thus, we can essentially characterize all FIR PR filter banks just by characterizing all causal FIR matrices with anticausal FIR inverses (abbreviated *cafacafi*).

In contrast, the family of causal FIR transfer matrices with causal FIR inverses (i.e., unimodlar matrices in z^{-1}) are not very useful in characterizing the class of all FIR PR filter banks. First, restricting the

† While it is true that an anticausal IIR filter $\mathbf{G}^{-1}(z)$ with infinitely long input is in general unrealizable, it becomes realizable if its input is the output of its causal inverse $\mathbf{G}(z)$.

polyphase matrix to be unimodular results in a loss of generality; given a causal FIR system with arbitrary FIR inverse, we cannot in general multiply it with a delay z^{-I} to obtain a causal FIR system with a causal FIR inverse. Furthermore, as shown in Sec. 2.1 of [1], unimodular matrices cannot in general be factorized into degree-one unimodular building blocks.

A subclass of FIR PR systems are FIR paraunitary filter banks (which correspond to orthonormal filter banks) where $\mathbf{E}(z)$ is unitary on the unit circle, and $\mathbf{E}^{\dagger}(1/z^*)\mathbf{E}(z) = \mathbf{I}$ (superscript dagger denoting transpose conjugation). For these systems, complete factorizations and characterizations have been found. See [8] for detailed discussions. In this case, the choice

$$\mathbf{R}(z) = \mathbf{E}^{-1}(z) = \mathbf{E}^{\dagger}(1/z^*) \tag{1.5}$$

guarantees perfection reconstruction. Eqn. (1.5) shows that if $\mathbf{E}(z)$ is causal FIR the inverse is anticausal FIR. In terms of the coefficients of the analysis filters $h_k(n)$ and synthesis filters $f_k(n)$, (1.5) is equivalent to the condition $f_k(n) = h_k^*(-n)$. In the IIR case this implies that if the analysis filters are causal and stable, the synthesis filters are either anticausal or unstable. In [16]-[18], techniques for implementing stable anticausal inverses are discussed.

The philosophy in this paper and the companion paper [1] is that by studying the more general class of *cafacafi* systems, of which paraunitary systems are special cases, we can characterize all FIR biorthonormal filter banks (with identical decimation ratio M in all channels). Study of *cafacafi* systems is more tractable than arbitrary FIR systems with FIR inverse, but at the same time it leads to very little loss of generality as we will show. As stated above, paraunitary systems are already special cases of *cafacafi* systems. One outcome of the proposed theory is the generalization of the lapped orthogonal transform (*LOT*) [21],[22],[3] to the biorthonormal case. This will be called the biorthonormal lapped transform (*BOLT*). In [1] we will present a factorization theorem that covers all *BOLT*s, and generates the *LOT* as a special case.

1.2. Main results and paper outline

1. In Sec. 2 we introduce causal systems with anticausal inverses. In the FIR case, the implementation of the anticausal inverse is trivial as long as we permit a finite latency between the input and the output (in this sense it is not really anticausal!). In Sec. 3 we show that even in the IIR case anticausal inverses can be implemented using the notion of state space descriptions. It is shown that such inversion is possible as long as we initialize the state variable of the inverse system properly. The result holds even with IIR input signals, and can be extended to time varying filter banks.
2. In Sec. 4 we study FIR systems with FIR inverses in terms of the Smith-form and Smith-McMillan form, which are well-known tools in system theory.

3. In Sec. 5 we study deeper the properties of linear time invariant systems with anticausal inverses. We show that an anticausal inverse exists if and *only if* the so-called *realization matrix* of a minimal implementation is nonsingular (Theorem 5.1). We then specialize to causal FIR systems with anticausal FIR inverses (*cafacafi* systems), and state the *cafacafi* property in terms of the Smith-McMillan form. We finally show in this section that for a causal FIR system $\mathbf{G}(z)$ having an FIR inverse, the inverse is anticausal if and only if the degree of [det $\mathbf{G}(z)$] is equal to the McMillan degree of $\mathbf{G}(z)$.

Notations and acronyms

1. Bold faced quantities (and calligraphic characters such as $\mathcal{R}, \mathcal{U}, \mathcal{V}$) represent matrices and vectors. The notations $\mathbf{A}^T, \mathbf{A}^*$ and $\mathbf{A}^\dagger$ represent, respectively, the transpose, conjugate, and transpose-conjugate of $\mathbf{A}$. The accent 'tilde' is defined as follows: $\widetilde{\mathbf{H}}(z) = \mathbf{H}^\dagger(1/z^*)$; thus if $\mathbf{H}(z) = \sum_n \mathbf{h}(n)z^{-n}$ then $\widetilde{\mathbf{H}}(z) = \sum_n \mathbf{h}^\dagger(-n)z^{-n}$. On the unit circle $\widetilde{\mathbf{H}}(z) = \mathbf{H}^\dagger(z)$.
2. The M-fold decimator ($\downarrow M$) has input output relation $y(n) = x(Mn)$, and for the expander ($\uparrow M$) it is $y(n) = x(n/M)$ when n = integer multiple of M, and zero otherwise [2], [8].
3. An FIR filter bank is one for which all the analysis *and* synthesis filters are FIR. Equivalently the matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ in Fig. 1.1 are FIR.
4. *Causality and anticausality.* A signal $\mathbf{x}(n)$ is causal if it is zero for $n < 0$ and anticausal if it is zero for $n > 0$. In both cases the signal could be nonzero for $n = 0$. Causal and anticausal LTI systems have impulse responses that are causal and anticausal respectively (see (1.2) and (1.3)). The output $\mathbf{y}(n)$ of an anticausal system depends only on the input $\mathbf{x}(m)$ for $m \geq n$.
5. *Cafacafi systems.* The phrase *CAusal Fir system with AntiCAusal Fir Inverse* arises many times in this paper, and will be abbreviated as *cafafaci*.
6. *Order versus degree.* The order of a causal rational transfer matrix $\mathbf{G}(z)$ is defined as the largest power of z^{-1} in its expression, whereas the McMillan degree (often called just *degree*) is the smallest number of delays with which we can implement the system. For example if $\mathbf{G}(z) = \mathbf{g}(0) + z^{-1}\mathbf{g}(1)$ with $\mathbf{g}(1) \neq \mathbf{0}$ then its order = 1 whereas the degree equals the rank of $\mathbf{g}(1)$ (p. 667 of [8]). For anticausal systems we define the order and degree in a similar way. For example, the degree is the minimum number of advance operators (z elements) required to implement the system.
7. The ideal time-reversal operator TR (Fig. 1.3(a)) has the input output relation $y(n) = x(-n)$. If we sandwich an LTI system with transfer function $H(z)$ between two TR operators, the result remains LTI with transfer function $H(z^{-1})$ (Fig. 1.3(b)). So if $H(z)$ is causal, the system in Fig. 1.3(b) is anticausal.

2. CAUSAL LTI SYSTEMS WITH ANTICAUSAL INVERSES

An r-input p-output LTI system is characterized by a $p \times r$ transfer matrix $\mathbf{G}(z)$. It has an inverse (left inverse to be precise) if there exits $\mathbf{H}(z)$ such that $\mathbf{H}(z)\mathbf{G}(z) = \mathbf{I}_r$. If $p = r$ then the inverse $\mathbf{H}(z)$, if it exists, is unique in the z-domain. However, the inverse z-transform of $\mathbf{H}(z)$ may still not be unique.

Consider the scalar example $G(z) = 1 - az^{-1}$. The inverse is $H(z) = 1/(1 - az^{-1})$, and has the causal impulse response $h(n) = a^n 1(n)$ (where $1(n)$ is the unit step) or the anticausal impulse response $h(n) = -a^n 1(-n-1)$ depending on the region of convergence chosen for the z-transform [23]. Thus an anticausal inverse exists in this case, even though there also exists a causal inverse. Unless $a = 0$ only one of these inverses is stable.

In the above scalar example, the system $G(z)$ is FIR and the inverse is IIR. In the matrix case, it is possible to have nontrivial examples of FIR matrices with FIR inverses. Here are three possible situations:

Example 2.1: Causal FIR with causal FIR inverse (unimodular matrix in z^{-1}):

$$\mathbf{G}(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \mathbf{G}^{-1}(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - z^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{2.1}$$

Example 2.2: Causal FIR with anticausal FIR inverse (*cafacafi*):

$$\mathbf{G}(z) = 0.5 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 0.5z^{-1} \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}, \qquad \mathbf{G}^{-1}(z) = 0.5 \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 0.5z \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \tag{2.2}$$

Example 2.3: Causal FIR with mixed FIR inverse:

$$\begin{aligned} \mathbf{G}(z) &= \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + z^{-1} \begin{bmatrix} 0 & 1 \\ 2 & -1 \end{bmatrix} + z^{-2} \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix} \\ \mathbf{G}^{-1}(z) &= 0.25 \begin{bmatrix} 1 & 1 \\ 2 & 0 \end{bmatrix} - 0.25z^{-1} \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} + 0.25z \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \end{aligned} \tag{2.3}$$

In each case the inverse is unique in the z domain (since $p = r = 2$) as well as in the time domain (since the inverse is FIR). In all the examples, $\mathbf{G}(z)$ is causal FIR and the inverse is FIR.

Some known IIR anticausal filter banks

There is a class of two-channel IIR filter banks called power symmetric filter banks (Sec. 5.3 of [8]). Here the two analysis filters have the form

$$H_0(z) = \frac{a_0(z^2) + z^{-1}a_1(z^2)}{2}, \qquad H_1(z) = \frac{a_0(z^2) - z^{-1}a_1(z^2)}{2} \tag{2.4}$$

where $a_0(z)$ and $a_1(z)$ are stable allpass functions. If the synthesis filters are chosen as $F_0(z) = H_0(z)$ and $F_1(z) = -H_1(z)$ then the analysis/synthesis system is free from aliasing and amplitude distortions, and suffers only from phase distortion. An example of $H_0(z)$ satisfying the above form is a digital Butterworth or elliptic lowpass filter with specifications satisfying the power symmetric condition (p. 211 of [8]). Fig. 2.1(a) shows the polyphase implementation of this system.

It was proposed in [14] that if the synthesis bank is chosen as in Fig. 2.1(b), that is, the synthesis filters are chosen as

$$F_0(z) = \widetilde{a}_0(z^2) + z\widetilde{a}_1(z^2), \qquad F_1(z) = \widetilde{a}_0(z^2) - z\widetilde{a}_1(z^2), \tag{2.5}$$

then the system will have perfect reconstruction, i.e., the phase distortion mentioned above will also be eliminated. This follows from the fact that an allpass function $a_i(z)$ satisfies $\widetilde{a}_i(z)a_i(z) = 1$. But if $a_i(z)$ are causal stable allpass filters they have poles inside the unit circle and zeros outside. So $\widetilde{a}_i(z)$ has all poles outside the unit circle making them unstable unless the filters are implemented in an anticausal manner. It was shown in [16] that such an anticausal synthesis bank can indeed be implemented provided we appropriately transmit the state variables of the filter realizations in the analysis bank. In Sec. 3 we will present this in a more general context for arbitrary linear systems using the state space formulation.

3. IMPLEMENTATION OF ANTICAUSAL INVERSES

Consider an M-input M-output causal system with $M \times M$ transfer matrix $\mathbf{G}(z)$, and let the state space description of a minimal implementation be

$$\begin{bmatrix} \mathbf{x}(n+1) \\ \mathbf{y}(n) \end{bmatrix} = \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}}_{\mathcal{R}} \begin{bmatrix} \mathbf{x}(n) \\ \mathbf{u}(n) \end{bmatrix} \tag{3.1}$$

so that $\mathbf{G}(z) = \mathbf{D} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$. We will assume that $[\det \mathbf{G}(z)] \not\equiv 0$ so that a unique inverse $\mathbf{G}^{-1}(z)$ exists. If this has an anticausal inverse z-transform, then we say that an anticausal inverse of $\mathbf{G}(z)$ exists. In general this is not guaranteed (even if $[\det \mathbf{G}(z)] \not\equiv 0$). For example, $\begin{bmatrix} 1 & 0 \\ z^{-1} & 1 \end{bmatrix}$ has the unique causal inverse $\mathbf{H}(z) = \begin{bmatrix} 1 & 0 \\ -z^{-1} & 1 \end{bmatrix}$, and there does not exist an anticausal inverse. In Sec. 5.1 and 5.2 we provide necessary and sufficient conditions for the existence of anticausal inverses.

3.1. Finding and implementing an anticausal inverse

The matrix $\mathcal{R}$ in (3.1) is said to be the *realization matrix* of the implementation. If this is nonsingular, we can find an anticausal inverse, as we now show. † For this consider the causal system described by

$$\begin{bmatrix} \widehat{\mathbf{x}}(n+1) \\ \widehat{\mathbf{y}}(n) \end{bmatrix} = \begin{bmatrix} \widehat{\mathbf{A}} & \widehat{\mathbf{B}} \\ \widehat{\mathbf{C}} & \widehat{\mathbf{D}} \end{bmatrix} \begin{bmatrix} \widehat{\mathbf{x}}(n) \\ \widehat{\mathbf{u}}(n) \end{bmatrix} \tag{3.2}$$

where

$$\begin{bmatrix} \widehat{\mathbf{A}} & \widehat{\mathbf{B}} \\ \widehat{\mathbf{C}} & \widehat{\mathbf{D}} \end{bmatrix} \triangleq \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \mathcal{R}^{-1} \tag{3.3}$$

† We will show later (Theorem 5.1) that if $\mathcal{R}$ is singular, then an anticausal inverse will not even exist.

The input and output of this system are denoted $\hat{\mathbf{u}}(n)$ and $\hat{\mathbf{y}}(n)$ respectively, to distinguish them from (3.1). To find its transfer function in terms of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, note that if we premultiply (3.2) with $\mathcal{R}$ and take z-transforms we can eliminate the state vector $\hat{\mathbf{x}}(n)$ to obtain

$$\hat{\mathbf{U}}(z) = \underbrace{\left(\mathbf{C}\left(z^{-1}\mathbf{I} - \mathbf{A}\right)^{-1}\mathbf{B} + \mathbf{D}\right)}_{\mathbf{G}(z^{-1})}\hat{\mathbf{Y}}(z)$$

so that $\hat{\mathbf{Y}}(z) = [\mathbf{G}(z^{-1})]^{-1}\hat{\mathbf{U}}(z)$. In other words, the transfer function of the causal system (3.2) is given by $\mathbf{H}(z^{-1}) \triangleq \left[\mathbf{G}(z^{-1})\right]^{-1}$. This has a causal impulse response $\mathbf{h}(-n)$. Now consider the scheme of Fig. 3.1(a). Here, the causal system (3.2) is sandwiched between the time-reversal (TR) operators. It therefore has the transfer function $\mathbf{H}(z) = \left[\mathbf{G}(z)\right]^{-1}$ indeed (compare with Fig. 1.3(b)), and its *anticausal* impulse response is $\mathbf{h}(n)$. Fig. 3.1(b) shows an equivalent representation of this system, where we have used $z\mathbf{I}$ instead of $z^{-1}\mathbf{I}$, thereby eliminating the TR operators. Finally Fig. 3.1(c) shows the internal details of the system of Fig. 3.1(a).

Transfer function, poles and eigenvalues. From Fig. 3.1(b) we see that the transfer function of the inverse can be written as

$$\mathbf{G}^{-1}(z) = \hat{\mathbf{D}} + \hat{\mathbf{C}}(z^{-1}\mathbf{I} - \hat{\mathbf{A}})^{-1}\hat{\mathbf{B}} \tag{3.4}$$

which should be compared with the transfer function of (3.1) which is $\mathbf{G}(z) = \mathbf{D} + \mathbf{C}(z\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}$. The eigenvalues of $\mathbf{A}$ are the poles of $\mathbf{G}(z)$ [31], [8], whereas the eigenvalues of $\hat{\mathbf{A}}$ are the reciprocals of the poles of $\mathbf{G}^{-1}(z)$. If $\mathbf{G}^{-1}(z)$ is anticausal stable, then the poles are outside the unit circle so that the eigenvalues of $\hat{\mathbf{A}}$ are *inside* the unit circle.

3.2. Choice of initial conditions

If we apply an input $\mathbf{u}(n)$ to the system (3.1) under zero initial conditions, then the output $\mathbf{y}(n)$ is possibly of infinite duration even if the input might be of finite duration (FIR). In theory, if this infinite-length output $\mathbf{y}(n)$ is "fed" into the system in Fig. 3.1(a), its output will be $\mathbf{u}(n)$. For, we have shown the transfer function of Fig. 3.1(a) to be the inverse of that of (3.1). In practice this requires infinite latency (or infinite storage) because of the idealized time reversal operators operating on possibly infinitely long signals.

In practice we can reduce this latency to a finite value by using the side information provided by the state vector $\mathbf{x}(n)$. This is achieved by performing the computation in blocks. We will explain the details by refering to the state space equations (3.1) and (3.2). Suppose we start the system (3.1) with the initial state $\mathbf{x}(0)$ and apply the causal input $\mathbf{u}(n)$, possibly of infinite duration. Consider a segment of L samples

$$\mathbf{u}(0), \mathbf{u}(1), \ldots, \mathbf{u}(L-1) \tag{3.5}$$

where L is an arbitrary integer. Denote the output during this period as

$$\mathbf{y}(0), \mathbf{y}(1), \ldots, \mathbf{y}(L-1). \tag{3.6}$$

The state vector $\mathbf{x}(L)$ and the above segment of the output are completely determined by the input segment (3.5) and initial state $\mathbf{x}(0)$. Based only on the knowledge of $\mathbf{x}(L)$ and the above finite segment of the output we can reconstruct the input segment (3.5) and the initial state $\mathbf{x}(0)$, if $\mathcal{R}$ in (3.1) is nonsingular. For example, from the knowledge of $\mathbf{x}(L)$ and $\mathbf{y}(L-1)$ we can compute

$$\begin{bmatrix} \mathbf{x}(L-1) \\ \mathbf{u}(L-1) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(L) \\ \mathbf{y}(L-1) \end{bmatrix} \tag{3.7}$$

and from the knowledge of $\mathbf{x}(L-1)$ and $\mathbf{y}(L-2)$ we can then compute

$$\begin{bmatrix} \mathbf{x}(L-2) \\ \mathbf{u}(L-2) \end{bmatrix} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{x}(L-1) \\ \mathbf{y}(L-2) \end{bmatrix} \tag{3.8}$$

and so forth. More generally, if we run the state space equations (3.2) by setting the initial state to be

$$\widehat{\mathbf{x}}(L) = \mathbf{x}(L), \tag{3.9}$$

and the input to be

$$\widehat{\mathbf{u}}(L+k) = \mathbf{y}(L-1-k), \qquad 0 \le k \le L-1 \tag{3.10}$$

the output for this duration will be

$$\widehat{\mathbf{y}}(L+k) = \mathbf{u}(L-1-k), \quad 0 \le k \le L-1, \tag{3.11}$$

and the final state will be $\widehat{\mathbf{x}}(2L) = \mathbf{x}(0)$. This is schematically shown in Fig. 3.2. Since $\widehat{\mathbf{y}}(2L-1) = \mathbf{u}(0)$ we see that the latency is equal to $2L-1$.

Summarizing, if we know a finite-duration segment of the output $\mathbf{y}(n)$ of the system (3.1), and the state variable $\mathbf{x}(L)$ at the end of this duration, we can use the above time-reversed segment and the state $\mathbf{x}(L)$ to compute the corresponding input segment (3.5) and the initial state $\mathbf{x}(0)$. We can repeat this process for the next set of L input samples

$$\mathbf{u}(L), \mathbf{u}(L+1), \ldots, \mathbf{u}(2L-1). \tag{3.12}$$

If the output of (3.1) in response to the above input and initial state $\mathbf{x}(L)$ is

$$\mathbf{y}(L), \mathbf{y}(L+1), \ldots, \mathbf{y}(2L-1), \tag{3.13}$$

we can recompute the input segment (3.12) and the initial state $\mathbf{x}(L)$ from the output segment (3.13) and the final state $\mathbf{x}(2L)$.

This time-reversed inversion process can be repeated indefinitely no matter how long the input signal is, simply by working with blocks of length L. In effect the ideal unrealizable time reversal operator in Fig. 3.1(a) has been replaced with (3.11) which represents the time reversal of a finite-size block. In order to perform the inversion (i.e., compute the input $\mathbf{u}(n)$ from the output $\mathbf{y}(n)$, the inversion process needs the "side information" in the form of the state vectors

$$\mathbf{x}(L), \mathbf{x}(2L), \mathbf{x}(3L), \ldots \tag{3.14}$$

This is the L-fold decimated version $\{\mathbf{x}(nL)\}$ of the state vector sequence $\{\mathbf{x}(n)\}$. As the block length L gets smaller, the latency or delay in the inversion process gets smaller, but the required amount of "side information" per unit time increases.

In subband coding practice, the subband signals $\mathbf{y}(n)$ are (heavily) quantized, and the above inversion cannot reconstruct $\mathbf{u}(n)$ perfectly even if $\mathbf{x}(nL)$ has high precision. In fact, one can solve for the best choice of initializing state vectors, that minimize the mean square reconstruction error in presence of subband quantization [24]. This optimal state sequence $\{\mathbf{x}_o(nL)\}$ can be transmitted (instead of $\{\mathbf{x}(nL)\}$) with high precision, and used as the side information for signal reconstruction.

Transmission of the state

Returning to the scheme of Eq. (3.9)-(3.11), suppose we set the initial state to $\widehat{\mathbf{x}}(L) = \mathbf{0}$ instead of $\widehat{\mathbf{x}}(L) = \mathbf{x}(L)$, and apply the input (3.10). Then by linearity the final state will be

$$\widehat{\mathbf{x}}(2L) = \mathbf{x}(0) - \widehat{\mathbf{A}}^L \mathbf{x}(L), \tag{3.15}$$

instead of $\widehat{\mathbf{x}}(2L) = \mathbf{x}(0)$. Thus, if $\mathbf{x}(L)$ is not transmitted, we can estimate it by computing $\widehat{\mathbf{x}}(2L)$ as above and solving for $\mathbf{x}(L)$. In the context of filter bank implementation, this means that there is no need to transmit the state $\mathbf{x}(L)$ because it can be estimated from the subband outputs (3.6) provided $\mathbf{x}(0)$ is known. Usually $\mathbf{x}(0) = \mathbf{0}$ so this is not a limitation. However, this approach to estimating the initial state at the synthesis bank end (rather than transmitting it) has some limitations.

First, in the case of IIR inputs where we have to transmit the states and outputs in blocks, it is not appropriate to assume $\mathbf{x}(0) = \mathbf{0}$ at the beginning of each block. So the above alternative reduces to transmitting the initial state rather than the final state periodically, and this does not save us anything. Second, the above estimation of $\mathbf{x}(L)$ involves inversion of $\widehat{\mathbf{A}}$, and fails if this matrix is singular. Finally, since the subbands are usually quantized heavily (i.e., (3.6) are quantized), we get $\widehat{\mathbf{x}}(2L) = \mathbf{x}(0) - \widehat{\mathbf{A}}^L \mathbf{x}(L) +$ error, and the estimation of $\mathbf{x}(L)$ by inversion of $\widehat{\mathbf{A}}^L$ might further amplify this error. In fact, the motivation for time reversed implementation of IIR inverses came from the fact that certain synthesis filters were stable only in the anticausal form [14]. In these cases the eigenvalues of $\widehat{\mathbf{A}}$ are inside the unit circle (see end of Sec. 3.1) so that it is not wise to compute $\widehat{\mathbf{A}}^{-L}$ for large L.

The best strategy therefore is to trasmit the side information (final state at the end of each block) rather than trying to estimate it from the quantized subbands. The increase of data rate due to this side information is negligible when the block length is large.

Generalizations

1. *The rectangular case.* It can be shown that the above time-reversal technique works for the case of $p \times r$ transfer matrices (as arises in nonmaximally decimated filter banks) provided the matrix $\mathcal{R}$, which is now rectangular, has a left inverse [25]. Details are omitted here.
2. *The time-varying case.* Recently there has been some interest in the design and implementation of time varying filter banks with the perfect reconstruction property [25]–[29]. In this case the polyphase matrices $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are replaced with time varying linear systems. The state space equations (3.1) and (3.2) are accordingly time varying, that is $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ are replaced with $\mathbf{A}(n)$, $\mathbf{B}(n)$, $\mathbf{C}(n)$ and $\mathbf{D}(n)$, and the realization matrix becomes a function of time, $\mathcal{R}(n)$. If $\mathcal{R}(n)$ is nonsingular for each n, then the time-reversed inversion process described previously continues to work with slight change of notations.

4. FIR SYSTEMS WITH FIR INVERSES

We now state some preliminary results for causal FIR transfer matrices with FIR inverses, paving the way for more results in the following sections. In all discussions, "causal FIR" is equivalent to "polynomial in z^{-1}" and "anticausal FIR" is equivalent to "polynomial in z."

Unimodular matrices. A unimodular matrix $\mathbf{U}(z)$ in z is a polynomial matrix in z, with the property [det $\mathbf{U}(z)$] = nonzero constant [30],[31],[8]. Note that

1. $\mathbf{U}^{-1}(z)$ exists and is also unimodular in z. So $\mathbf{U}(z)$ is anticausal FIR with anticausal FIR inverse.
2. None of the columns (or rows) of $\mathbf{U}(z)$ has a factor $f(z)$ other than a constant. (Otherwise [det $\mathbf{U}(z)$] would have this factor, which is not possible.)
3. We can write $\mathbf{U}(z) = \mathbf{u}(0) + z\mathbf{u}(1) + z^2\mathbf{u}(2)\ldots$ Note that $\mathbf{U}(0) = \mathbf{u}(0)$ so [det $\mathbf{u}(0)$] = [det $\mathbf{U}(0)$] $\neq 0$, that is $\mathbf{u}(0)$ is nonsingular. In particular, therefore, $\mathbf{u}(0) \neq \mathbf{0}$.
4. A unimodular matrix in z^{-1} is a polynomial matrix in z^{-1} with the above properties. It is a causal FIR system with a causal FIR inverse. For an example, see beginning of Sec. 2.

For the rest of this section $\mathbf{G}(z)$ is a $p \times r$ matrix with normal rank r (i.e., there is some z for which the rank is r). Note that $\mathbf{G}(z)$ has r inputs and p outputs. In maximally decimated filter banks, the polyphase matrices satisfy $p = r$, but we allow $p \neq r$ to permit non maximally decimated systems ($p > r$). Such systems also find applications in the theory of convolutional codes [32]. Unless mentioned otherwise, 'inverse' stands for left inverse. Thus $\mathbf{H}(z)$ is an inverse if $\mathbf{H}(z)\mathbf{G}(z) = \mathbf{I}_r$.

4.1. The Smith-form and the Smith-McMillan form [33],[30],[31],[8]

Given a $p \times r$ polynomial matrix $\mathbf{P}(x)$ in the variable x, it can always be expressed in the form $\mathbf{P}(x) = \mathbf{U}(x)\mathbf{\Gamma}(x)\mathbf{W}(x)$ where (i) $\mathbf{U}(x)$ and $\mathbf{W}(x)$ are unimodular in x and (ii) $\mathbf{\Gamma}(x)$ is a $p \times r$ diagonal matrix with the first ρ diagonal elements $\gamma_i(x), 0 \leq i \leq \rho - 1$ that are polynomials in x. (This is the Smith decomposition, known for over hundred years [33]). Here ρ is the normal rank of $\mathbf{P}(x)$. The remaining diagonal elements of $\mathbf{\Gamma}(x)$ are zero. The polynomials $\gamma_i(x)$ can always be assumed to be monic (i.e., highest power of x has coefficient unity) and furthermore $\gamma_i(x)|\gamma_{i+1}(x)$, that is, $\gamma_i(x)$ is a factor of $\gamma_{i+1}(x)$. Such a matrix $\mathbf{\Gamma}(x)$ is said to be the Smith-form of $\mathbf{P}(x)$, and is unique (but $\mathbf{U}(x)$ and $\mathbf{W}(x)$ are not). In this paper we will have occasions to use the Smith-form of polynomials in z (anticausal FIR systems) as well as polynomials in z^{-1} (causal FIR).

The Smith-McMillan form, which derives from the Smith-form, is defined only for causal rational systems. Thus let $\mathbf{G}(z)$ be a $p \times r$ matrix of rational functions (ratios of polynomials in z or z^{-1}) representing a causal system. We first write $\mathbf{G}(z) = \mathbf{G}_1(z)/d(z)$ where $d(z)$ is a polynomial in z of sufficiently high order that all the elements of $\mathbf{G}_1(z)$ are polynomials in z. We then express $\mathbf{G}_1(z)$ in Smith-form $\mathbf{U}(z)\mathbf{\Gamma}(z)\mathbf{W}(z)$ (all quantities are polynomials in z) and then divide the diagonal elements of $\mathbf{\Gamma}(z)$ by $d(z)$ to obtain the form $\mathbf{G}(z) = \mathbf{U}(z)\mathbf{\Lambda}(z)\mathbf{W}(z)$. Here $\mathbf{U}(z)$ and $\mathbf{W}(z)$ are unimodular polynomials in z and $\mathbf{\Gamma}(z)$ is a $p \times r$ diagonal matrix with the first ρ diagonal elements $\lambda_i(z) = \alpha_i(z)/\beta_i(z)$. In this scheme $\alpha_i(z)$ and $\beta_i(z)$ are polynomials in z with no common factors for a given i, and we have $\alpha_i(z)|\alpha_{i+1}(z)$, and $\beta_{i+1}(z)|\beta_i(z)$. The sum of orders of all the $\beta_i(z)$ polynomials can be shown to be equal to the McMillan degree of the causal rational system $\mathbf{G}(z)$.

The Smith form and the Smith-McMillan form are covered in many references [30]–[32]. We will use some of the properties given in these references. A review can be found in Sec. 13.5 of [8].

Theorem 4.1. Let $\mathbf{G}(z)$ be $p \times r$ *causal FIR* with normal rank r. Consider the Smith decomposition $\mathbf{G}(z) = \mathbf{U}(z)\mathbf{\Gamma}(z)\mathbf{W}(z)$ where $\mathbf{U}(z)$ and $\mathbf{W}(z)$ are unimodular in z^{-1}. (Since $\mathbf{G}(z)$ is causal FIR, the diagonal elements $\gamma_i(z)$ of $\mathbf{\Gamma}(z)$ are causal FIR. Also $\gamma_i(z) \not\equiv 0$ for $0 \leq i \leq r-1$ since normal rank $= r$.) Then

1. $\mathbf{G}(z)$ has an *FIR inverse* if and only if $\gamma_i(z) = z^{-n_i}$ and $n_i \geq 0$ for $0 \leq i \leq r-1$.
2. $\mathbf{G}(z)$ has a *causal FIR inverse* if and only if we can write $\gamma_i(z) = 1$ for $0 \leq i \leq r-1$. $\diamond$

Proof of Part 1. Since $p \neq r$ in general, the proof is somewhat tricky. If $\gamma_i(z) = z^{-n_i}$, we can take the inverse to be $\mathbf{W}^{-1}(z)\mathbf{\Gamma}^{-1}(z)\mathbf{U}^{-1}(z)$ where $\mathbf{\Gamma}^{-1}(z)$ is the left inverse of the $p \times r$ matrix $\mathbf{\Gamma}(z)$ (obtained by replacing $\gamma_i(z)$ with $1/\gamma_i(z)$, and transposing), and we are done. Conversely, suppose there is an FIR inverse $\mathbf{H}(z)$. Apply an input $\mathbf{X}(z)$ to the system $\mathbf{G}(z)$ such that

$$\mathbf{W}(z)\mathbf{X}(z) = \begin{bmatrix} 1/\gamma_0(z) \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \text{i.e.,} \quad \mathbf{X}(z) = \frac{\widehat{\mathbf{W}}_0(z)}{\gamma_0(z)} \tag{4.1}$$

where $\widehat{\mathbf{W}}_0(z)$ is the 0th column of $\widehat{\mathbf{W}}(z) \stackrel{\Delta}{=} \mathbf{W}^{-1}(z)$. Then the output is

$$\mathbf{Y}(z) = \mathbf{U}(z)\mathbf{\Gamma}(z)\mathbf{W}(z)\mathbf{X}(z) = \mathbf{U}_0(z), \tag{4.2}$$

where $\mathbf{U}_0(z)$ is the 0th column of $\mathbf{U}(z)$. Unless $\gamma_0(z)$ has the form z^{-n_0}, the above input $\mathbf{X}(z)$ is IIR [because $\mathbf{W}^{-1}(z)$ is unimodular and its 0th column $\widehat{\mathbf{W}}_0(z)$ cannot have the common factor $\gamma_0(z)$]. The output $\mathbf{U}_0(z)$ is however FIR (Fig. 4.1). Thus the FIR inverse system $\mathbf{H}(z)$ produces IIR *output* $\widehat{\mathbf{W}}_0(z)/\gamma_0(z)$ in response to FIR *input* $\mathbf{U}_0(z)$, and this a contradiction. So we conclude $\gamma_i(z)$ has the form z^{-n_i}. That $n_i \geq 0$ follows from the fact that the Smith form is also causal.

Proof of Part 2. Again, if $\gamma_i(z) = 1$, then the inverse $\mathbf{W}^{-1}(z)\mathbf{\Gamma}^{-1}(z)\mathbf{U}^{-1}(z)$ is causal FIR, and we are done. Consider the converse. We already showed that if there is an FIR inverse then $\gamma_i(z) = z^{-n_i}$, with $n_i \geq 0$. In the above input/output construction, the inverse system $\mathbf{H}(z)$ is such that the input $\mathbf{U}_0(z)$ produces the output $z^{n_0}\widehat{\mathbf{W}}_0(z)$. But since $\mathbf{U}_0(z)$ and $\widehat{\mathbf{W}}_0(z)$ are columns of unimodular matrices in z^{-1}, they have nonzero constant coefficients. Thus if $n_0 > 0$, the output of $\mathbf{H}(z)$ is noncausal in response to a causal input (Fig. 4.2). Since this violates causality of the inverse $\mathbf{H}(z)$, we conclude $n_0 = 0$, that is, $\gamma_0(z) = 1$. Similarly $\gamma_i(z) = 1$, for $0 \leq i \leq r-1$. $\nabla\nabla\nabla$

Theorem 4.2. Let $\mathbf{G}(z)$ be $p \times r$ causal FIR. Then $\mathbf{G}(z)$ has a *causal FIR inverse* if and only if it is a submatrix of a $p \times p$ unimodular matrix in z^{-1}. $\diamond$

Proof. Let $\mathbf{U}_1(z)$ be $p \times p$ unimodular in z^{-1} such that $\mathbf{G}(z)$ is the leftmost $p \times r$ submatrix. Then we can write $\mathbf{G}(z) = \mathbf{U}_1(z)\begin{bmatrix}\mathbf{I}_r \\ \mathbf{0}\end{bmatrix}$ so that $\mathbf{G}^{-1}(z) = [\mathbf{I}_r \quad \mathbf{0}]\,\mathbf{U}_1^{-1}(z)$. Thus $\mathbf{G}(z)$ has a causal FIR inverse. Conversely, suppose there is a causal FIR inverse. By Theorem 4.1 (part 2),

$$\mathbf{G}(z) = \mathbf{U}(z)\begin{bmatrix}\mathbf{I} \\ \mathbf{0}\end{bmatrix}\mathbf{W}(z) = \mathbf{U}(z)\begin{bmatrix}\mathbf{W}(z) \\ \mathbf{0}\end{bmatrix} \tag{4.3}$$

where $\mathbf{U}(z)$ and $\mathbf{W}(z)$ are unimodular in z^{-1}. Consider the product

$$\mathbf{U}(z)\begin{bmatrix}\mathbf{W}(z) & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{p-r}\end{bmatrix} \tag{4.4}$$

This is unimodular in z^{-1}, and its leftmost $p \times r$ submatrix is indeed $\mathbf{G}(z)$. $\nabla\nabla\nabla$

4.2. State space reprsentations

Consider a $p \times r$ causal FIR system

$$\mathbf{G}(z) = \mathbf{g}(0) + z^{-1}\mathbf{g}(1) + \ldots + z^{-K}\mathbf{g}(K), \tag{4.5}$$

with $\mathbf{g}(K) \neq \mathbf{0}$ so that the order is K. Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be a minimal realization of this system. Then all the eigenvalues of $\mathbf{A}$ (which are the poles of the FIR system) are zero, and $\mathbf{A}^N = \mathbf{0}$ where N is the degree of the system (i.e., $\mathbf{A}$ is $N \times N$). Evidently $N \geq K$. Since $\mathbf{g}(n) = \mathbf{C}\mathbf{A}^{n-1}\mathbf{B}$ for $n > 0$, it is immediate that $\mathbf{C}\mathbf{A}^K\mathbf{B} = \mathbf{0}$. But more is true. It is shown in p. 709 [8] that $\mathbf{C}\mathbf{A}^K = \mathbf{0}$ and $\mathbf{A}^K\mathbf{B} = \mathbf{0}$.

In fact we can prove the stronger result that $\mathbf{A}^K = \mathbf{0}$. For this note that $\mathbf{A}^K\mathbf{B} = \mathbf{0}$ implies

$$\mathbf{A}^K [\mathbf{B} \quad \mathbf{A}\mathbf{B} \quad \ldots \mathbf{A}^{K-1}\mathbf{B}] = \mathbf{0}. \tag{4.6}$$

By reachability of the minimal realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$, the matrix following $\mathbf{A}^K$ above has full row rank N. So it follows that $\mathbf{A}^K = \mathbf{0}$. This also verifies $\mathbf{A}^N = \mathbf{0}$ since $N \geq K$.

In the next section we will see (in the $p = r$ case) that if the system $\mathbf{G}(z)$ has an anticausal FIR inverse, then the realization matrix $\mathcal{R}$ [Eq. (3.1)] is invertible. Moreover the state transition matrix $\widehat{\mathbf{A}}$ of the inverse has all eigenvalues equal to zero, and $\widehat{\mathbf{A}}^N = \mathbf{0}$.

5. PROPERTIES OF SYSTEMS WITH ANTICAUSAL INVERSES

In this section we develop some properties of causal transfer matrices with anticausal inverses. The first pertains to the realization matrix $\mathcal{R}$ of a state-space description (Theorem 5.1). The second pertains to the Smith-McMillan form (Theorem 5.2) and the third to the McMillan degree (Theorem 5.3). These results are useful in the implementation and factorizations [1] of such systems.

5.1. Nonsingularity of the realization matrix

Theorem 5.1. *Existence of anticausal inverse.* Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be the state space description of a minimal realization of a causal system with $M \times M$ transfer matrix $\mathbf{G}(z)$. Then $\mathbf{G}(z)$ has an anticausal inverse if and only if the realization matrix $\mathcal{R} \triangleq \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$ is nonsingular. $\diamond$

Proof. We have already shown that if $\mathcal{R}$ is nonsingular, the system of Fig. 3.1(a) is the anticausal inverse of $\mathbf{G}(z)$ (even if $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is not minimal). We only have to show that if there exists an anticausal inverse, then $\mathcal{R}$ is necessarily nonsingular. The proof uses the minimality (i.e., reachability and observability) of the realization $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$.

Let N be the McMillan degree of $\mathbf{G}(z)$. Suppose we start the system (3.1) at time $n = -N$ with initial condition $\mathbf{x}(-N) = \mathbf{0}$. In view of reachability, we can always find an input sequence

$$\ldots \mathbf{0}, \mathbf{0}, \mathbf{u}(-N), \mathbf{u}(-N+1), \ldots, \mathbf{u}(-1) \tag{5.1}$$

such that the state vector $\mathbf{x}(0)$ has any value of our choice. Having done this we can still choose $\mathbf{u}(0)$ in any manner. Thus, we can always arrange the vector $\begin{bmatrix}\mathbf{x}(0)\\ \mathbf{u}(0)\end{bmatrix}$ to be anything of our choice. Now (3.1) implies

$$\begin{bmatrix}\mathbf{x}(1)\\ \mathbf{y}(0)\end{bmatrix} = \underbrace{\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}}_{\mathcal{R}} \begin{bmatrix}\mathbf{x}(0)\\ \mathbf{u}(0)\end{bmatrix}. \tag{5.2}$$

If $\mathcal{R}$ is singular, we can choose $\begin{bmatrix}\mathbf{x}(0)\\ \mathbf{u}(0)\end{bmatrix}$ to be a nonzero vector such that

$$\begin{bmatrix}\mathbf{x}(1)\\ \mathbf{y}(0)\end{bmatrix} = \begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix} \begin{bmatrix}\mathbf{x}(0)\\ \mathbf{u}(0)\end{bmatrix} = \begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix} \tag{5.3}$$

Thus $\mathbf{x}(1) = \mathbf{0}$ and $\mathbf{y}(0) = \mathbf{0}$. With $\mathbf{u}(n) = \mathbf{0}$ for $n > 0$, the values of $\mathbf{x}(n+1)$ and $\mathbf{y}(n)$ will therefore be zero for all $n \geq 0$. Summarizing, we can find an input sequence

$$\ldots \mathbf{0}, \mathbf{0}, \mathbf{u}(-N), \mathbf{u}(-N+1), \ldots, \mathbf{u}(-1), \mathbf{u}(0), \mathbf{0}, \mathbf{0}, \ldots \tag{5.4}$$

such that the output has the form

$$\ldots \mathbf{0}, \mathbf{0}, \mathbf{y}(-N), \mathbf{y}(-N+1), \ldots, \mathbf{y}(-1), \mathbf{0}, \mathbf{0}, \ldots \tag{5.5}$$

under zero initial conditions ($\mathbf{x}(-N) = \mathbf{0}$).

Now suppose there exists an inverse for $\mathbf{G}(z)$, with transfer function $\mathbf{H}(z)$. This inverse would produce the FIR output (5.4) in response to the FIR input (5.5). So we obtain the schematic shown in Fig. 5.1. This inverse $\mathbf{H}(z)$ cannot therefore be anticausal (see definition in Sec. 1.2), unless $\mathbf{u}(0) = \mathbf{0}$ in the above construction. But if $\mathbf{u}(0) = \mathbf{0}$ then $\mathbf{x}(0) \neq \mathbf{0}$ (otherwise $\begin{bmatrix}\mathbf{x}(0)\\ \mathbf{u}(0)\end{bmatrix}$ would become zero), and (5.3) implies $\mathbf{A}\mathbf{x}(0) = \mathbf{0}$ and $\mathbf{C}\mathbf{x}(0) = \mathbf{0}$. This violates the PBH condition [31],[8] for complete observability, i.e., violates minimality. Summarizing, if $\mathcal{R}$ is singular, then there cannot exist an anticausal inverse. $\nabla\nabla\nabla$

Example 5.1. Consider $\mathbf{G}(z) = \begin{bmatrix}1 & 0\\ z^{-1} & 1\end{bmatrix}$. Since the unique inverse $\mathbf{H}(z) = \begin{bmatrix}1 & 0\\ -z^{-1} & 1\end{bmatrix}$ is causal, there is no anticausal inverse for $\mathbf{G}(z)$. This leads us to conclude that the realization matrix $\mathcal{R}$ is singular. Indeed, the minimal realization of $\mathbf{G}(z)$ given by Fig. 5.2 has $\mathbf{A} = 0, \mathbf{B} = [1 \quad 0], \mathbf{C} = [0 \quad 1]^T, \mathbf{D} = \mathbf{I}_2$ so that $\mathcal{R}$ is singular.

Example 5.2. Let $\mathbf{G}(z)$ be $M \times M$ causal paraunitary. Then the inverse is $\widetilde{\mathbf{G}}(z)$ and is anticausal. Thus causal paraunitary systems always have anticausal inverses (both FIR and IIR cases). This is consistent with the fact that the $\mathcal{R}$-matrix in this case is unitary upto similarity [8].

Example 5.3. Let $G(z) = g(0) + g(1)z^{-1} + \ldots + g(N)z^{-N}$ (single-input single-output FIR). Then the anticausal inverse can be obtained by long division [23]. This is consistent with the fact that the direct form structure has a nonsingular realization matrix (see, e.g., p. 670 of [8]) as long as $g(N) \neq 0$.

Example 5.4. Consider a causal IIR filter with transfer function $G(z) = \sum_{n=0}^{N} p_n z^{-n} / [1 + \sum_{n=1}^{D} q_n z^{-n}]$ with $p_N \neq 0$, $q_D \neq 0$. This has McMillan degree $= \max(N, D)$. It can be shown that the realization matrix of the direct-form structure is nonsingular if and only if $N \geq D$. For example, if $G(z) = (1+0.5z^{-1})/(1+0.6z^{-1})$ then there exists an anticausal inverse (namely the anticausal inverse z-transform of $1/G(z)$ [23]) whereas if $G(z) = 1/(1+0.6z^{-1})$, then there is no anticausal inverse.

Further Observations on the Anticausal Inverse

1. *Consequences of Theorem 5.1.* If ever $\mathbf{G}(z)$ has an anticausal inverse, it can be implemented as in Fig. 3.1(a) because $\mathcal{R}$ is guaranteed nonsingular. No loss of generality is therefore encountered by restricting ourself to the scheme of Fig. 3.1(a). Furthermore, if $\mathcal{R}$ is singular, an anticausal inverse does not exist anyway, and we need not look for an implementation.

2. It is well-known that the realization matrix $\mathcal{R}$ is invertible if $\mathbf{D}$ and $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ are nonsingular (or if $\mathbf{A}$ and $\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ are nonsingular). Neither of these, however, is a necessary condition. For example, let $A = 0, B = 1, C = 1$ and $D = 0$ so that $G(z) = z^{-1}$. Then $\mathcal{R} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and is nonsingular; the anticausal inverse is $H(z) = z$.

3. Since any two minimal realizations $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ and $(\mathbf{A}_1, \mathbf{B}_1, \mathbf{C}_1, \mathbf{D})$ are related by a similarity transformation, their realization matrices are related as

$$\underbrace{\begin{bmatrix} \mathbf{A}_1 & \mathbf{B}_1 \\ \mathbf{C}_1 & \mathbf{D} \end{bmatrix}}_{\mathcal{R}_1} = \begin{bmatrix} \mathbf{T}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}}_{\mathcal{R}} \begin{bmatrix} \mathbf{T} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

   so that $\mathcal{R}$ is invertible if and only if $\mathcal{R}_1$ is.

4. *Minimality of inverse realization.* The minimality of $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ implies that the anticausal realization in Fig. 3.1(b) is also minimal (Appendix A). In this sense, we will say that the anticausal realization in Fig. 3.1(a) is minimal. The number of delay and advance elements required to implement the TR operators are obviously infinite and are not counted.

5. *Degree of the inverse.* Since $\widehat{A}$ [illegible] the same size as $\mathbf{A}$, the preceding minimality result also shows that the degree of the anticausal inverse $\mathbf{G}^{-1}(z)$ (in z) is the same as the degree of $\mathbf{G}(z)$ (in z^{-1}). It is

appealing to notice that a similar result is true for $M \times M$ causal systems with *causal* inverses also (p. 712 of [8]).

6. *Nonsingularity and minimality.* Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ be some realization of a causal system. If $\mathcal{R}$ is nonsingular and $\mathbf{A}$ has all eigenvalues equal to zero, then $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is a minimal realization. To see this assume the contrary, say the realization is not observable. Then there is $\mathbf{v} \neq \mathbf{0}$ such that $\mathbf{Av} = \mathbf{0}$ and $\mathbf{Cv} = \mathbf{0}$ (PBH test, [31], [8]). This means the vector $\begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix}$ annihilates $\mathcal{R}$ violating nonsingularity.

7. *Stability.* Stability of $\mathbf{G}(z)$ does not imply that of the inverse system, in general. For example, let $G(z) = 1 - az^{-1}$, which is stable for any a. Then the inverse $H(z) = 1/(1 - az^{-1})$ has the anticausal inverse transform $-a^n 1(-n-1)$ [23] (where $1(n)$ is the unit step). This inverse is stable if and only if $|a| > 1$.

5.2. Smith-McMillan form and McMillan degree

We now present some results on the Smith-McMillan forms of FIR systems with FIR anticausal inverses. This will also reveal an interesting property, namely that the McMillan degree is equal to the determinantal degree in the square-matrix case.

Theorem 5.2. *Smith-McMillan form.* Let $\mathbf{G}(z)$ be $p \times r$ causal FIR with normal rank r. Let $\Lambda(z)$ be the Smith-McMillan form of $\mathbf{G}(z)$. That is $\mathbf{G}(z) = \mathbf{U}(z)\Lambda(z)\mathbf{W}(z)$ where $\mathbf{U}(z)$ and $\mathbf{W}(z)$ are unimodular in z and $\lambda_i(z) = \alpha_i(z)/\beta_i(z)$ where $\alpha_i(z), \beta_i(z)$ are polynomials in z. Then

1. There exists an *FIR inverse* $\mathbf{H}(z)$ if and only if $\lambda_i(z) = z^{-\ell_i}$, for $0 \leq i \leq r-1$.

2. There exists an *anticausal FIR inverse* $\mathbf{H}(z)$ if and only if $\lambda_i(z) = z^{-\ell_i}$ with $\ell_i \geq 0$, for $0 \leq i \leq r-1$. $\diamond$

Proof. First note that the FIR nature of $\mathbf{G}(z)$ implies $\beta_i(z) = z^{n_i}$ for some integers n_i. (Just recall how the Smith-McMillan form is constructed by first obtaining the Smith form of a polynomial in z, then dividing ...) So $\lambda_i(z) = z^{-n_i} \times$ (polynomials in z). The full normal rank of $\mathbf{G}(z)$ means that none of the r diagonal elements of the $p \times r$ matrix $\Lambda(z)$ is zero.

The proof for Part 1 is similar to part 1 of Theorem 4.1. We only have to prove Part 2. If $\lambda_i(z) = z^{-\ell_i}$, $\ell_i \geq 0$, then $\mathbf{W}^{-1}(z)\Lambda^{-1}(z)\mathbf{U}^{-1}(z)$ is an anticausal FIR inverse. (Here $\Lambda^{-1}(z)$ is the $r \times p$ matrix whose diagonal elements are $1/\lambda_i(z)$). We only have to prove that if there is an anticausal inverse $\mathbf{H}(z)$, then $\ell_i \geq 0$. Assume the contrary, for example, let $\ell_0 < 0$. Apply an anticausal input $\mathbf{X}(z)$ to the system $\mathbf{G}(z)$ such that $\mathbf{W}(z)\mathbf{X}(z) = [1 \quad 0 \quad \ldots \quad 0]^T$. (For this, just choose $\mathbf{X}(z)$ to be the 0th column of $\mathbf{W}^{-1}(z)$.) Then the output is $\mathbf{Y}(z) = z^K\mathbf{U}_0(z)$ with $K = -\ell_0 > 0$. Here $\mathbf{U}_0(z)$ is the 0th column of $\mathbf{U}(z)$. So the input and output are

$$\begin{aligned} \mathbf{X}(z) &= \ldots + z^2\mathbf{x}(-2) + z\mathbf{x}(-1) + \mathbf{x}(0) \\ \mathbf{Y}(z) &= \ldots + z^{K+2}\mathbf{y}(-K-2) + z^{K+1}\mathbf{y}(-K-1) + z^K\mathbf{y}(-K) \end{aligned} \tag{5.6}$$

Since the columns of unimodular matrices cannot have constant term equal to zero, we have $\mathbf{x}(0) \neq \mathbf{0}$ and $\mathbf{y}(-K) \neq \mathbf{0}$.

The inverse system $\mathbf{H}(z)$ produces $\mathbf{X}(z)$ in response to $\mathbf{Y}(z)$. Since $\mathbf{x}(0) \neq \mathbf{0}$ and $K > 0$, this contradicts anticausality of $\mathbf{H}(z)$ (Fig. 5.3). The conclusion is that it is not possible to have $\ell_0 < 0$. Similarly it follows that $\ell_i \geq 0$ for all i. $\nabla\nabla\nabla$

Theorem 5.3. *McMillan degree of causal FIR systems which have anticausal FIR inverses.* Let $\mathbf{G}(z)$ be $M \times M$ causal FIR with FIR inverse. Then $[\det \mathbf{G}(z)] = cz^{-N}$ for some integer $N \geq 0$. Moreover

$$N \leq \text{ McMillan degree of } \mathbf{G}(z) \tag{5.7}$$

with equality if and only if $\mathbf{G}(z)$ has an anticausal FIR inverse. $\diamond$

Proof. Consider the Smith-McMillan form $\mathbf{G}(z) = \mathbf{U}(z)\mathbf{\Lambda}(z)\mathbf{W}(z)$. Since this has an FIR inverse, we know from Theorem 5.2 that the diagonal elements of $\mathbf{\Lambda}(z)$ are $z^{-\ell_i}$. So $[\det \mathbf{G}(z)] = cz^{-N}$ where $N = \sum_i \ell_i$. On the other hand the McMillan degree is $\sum_{\ell_i \geq 0} \ell_i$. Thus

$$N = \sum_i \ell_i \leq \sum_{\ell_i \geq 0} \ell_i = \text{ McMillan degree}$$

From Theorem 5.2 we know that $\ell_i \geq 0$ for all i if and only if there exists an anticausal FIR inverse. So the result follows. $\nabla\nabla\nabla$

Comments.

1. A generalization of Theorem 5.3 for the rectangular case ($p \neq r$) can be found in Appendix B.
2. It is well-known that if $\mathbf{G}(z)$ is a causal $M \times M$ FIR paraunitary system then its determinant is given by cz^{-N} where N is the McMillan degree of $\mathbf{G}(z)$. We now see that this same property is what characterizes any causal FIR system with anticausal FIR inverse. (Note that the inverse of the paraunitary system is $\widetilde{\mathbf{G}}(z)$, which is indeed anticausal.)
3. For any causal system $\mathbf{G}(z)$, it is well-known that the degree of $[\det \mathbf{G}(z)]$ cannot exceed the McMillan degree of $[\mathbf{G}(z)]$ (Sec. 13.8 of [8]). According to Theorem 5.3, the degree of $[\det \mathbf{G}(z)]$ has this *maximum* value *if and only if* the inverse is *anticausal.*
4. An example of a system not satisfying the requirements of Theorem 5.3 is when $\mathbf{G}(z)$ is unimodular in z^{-1}. In this case the degree of the determinant is zero regardless of the McMillan degree. The inverse

system in this case is causal. This is an extreme example where the degree of the determinant is the *smallest* possible, and the inverse is entirely *causal*.

5.3. Impulse response

Let $\mathbf{G}(z) = \sum_{n=0}^{K} z^{-n}\mathbf{g}(n)$ be $M \times M$, and let $\mathbf{H}(z) = \sum_{n=0}^{L} z^{n}\mathbf{h}(n)$ be the anticausal FIR inverse. Assume the orders $K, L > 0$ to avoid trivialities, and let $\mathbf{g}(0)$, $\mathbf{g}(K)$, $\mathbf{h}(0)$, and $\mathbf{h}(L)$ be nonnull matrices. Then all of these are singular. To see this, note that the property $\mathbf{G}(z)\mathbf{H}(z) = \mathbf{I}$ implies, among other things, the following:

$$\mathbf{g}(0)\mathbf{h}(L) = \mathbf{0}, \quad \text{and} \quad \mathbf{g}(K)\mathbf{h}(0) = \mathbf{0}, \tag{5.8}$$

so that all the four matrices are singular.

Now suppose that we are given some $M \times M$ causal FIR transfer matrix $\mathbf{G}(z) = \sum_{n=0}^{K} z^{-n}\mathbf{g}(n)$, $K > 0$, with an FIR inverse. If $\mathbf{g}(K)$ is nonsingular, then the inverse is guaranteed to be anticausal! (This does not violate (5.8) as $\mathbf{h}(0)$ is guaranteed to be zero; see below.) To see this define

$$\mathbf{F}(z) = z^{K}\mathbf{G}(z) = \mathbf{g}(K) + z\mathbf{g}(K-1) + \ldots + z^{K}\mathbf{g}(0). \tag{5.9}$$

If $\mathbf{g}(K)$ is nonsingular, then an anticausal inverse $\mathbf{F}^{-1}(z)$ (possibly IIR) will exist (use Sec. 13.10.1 of [8], with z^{-1} replaced by z everywhere). So $\mathbf{G}(z)$ has the anticausal inverse $\mathbf{H}(z) \triangleq z^{K}\mathbf{F}^{-1}(z)$. Since $\mathbf{F}^{-1}(z)$ is anticausal and $K > 0$, this means that $\mathbf{h}(n) = \mathbf{0}$, $n < K$.

The highest coefficient $\mathbf{g}(K)$ and the Smith-McMillan form

We can draw further interesting conclusions about the coefficient of the highest power of z^{-1}. Let $\mathbf{G}(z)$ be a causal FIR system with FIR inverse. Then the Smith McMillan form has diagonal elements $z^{-\ell_i}$ and we can assume $\ell_0 \geq \ell_1 \ldots$ (This follows from the divisibility properties $\alpha_i(z)|\alpha_{i+1}(z)$, and $\beta_{i+1}(z)|\beta_i(z)$). Suppose the first s diagonal elements are equal, that is $K = \ell_0 = \ell_1 \ldots = \ell_{s-1} > \ell_s$. Then

$$\mathbf{G}(z) = \underbrace{\left(\mathbf{u}_0 + z\mathbf{u}_1 + \ldots\right)}_{p\times p \ \mathbf{U}(z)} \underbrace{\begin{bmatrix} z^{-K}\mathbf{I}_s & \mathbf{0} \\ \mathbf{0} & \times \end{bmatrix}}_{p\times r \ \mathbf{\Lambda}(z),} \underbrace{\left(\mathbf{w}_0 + z\mathbf{w}_1 + \ldots\right)}_{r\times r \ \mathbf{W}(z)}$$

where the elements of $\times$ are FIR, with all powers z^{-i} satisfying $i < K$. Thus $\mathbf{G}(z) = \sum_{n=0}^{K} \mathbf{g}(n)z^{-n}$, with

$$\mathbf{g}(K) = \mathbf{u}_0 \begin{bmatrix} \mathbf{I}_s & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \mathbf{w}_0.$$

Since $\mathbf{u}_0$ and $\mathbf{w}_0$ are nonsingular, the rank of $\mathbf{g}(K)$ is equal to s. In particular $\mathbf{g}(K)$ has full rank ($= r$ since $p \geq r$) if and only if

$$\mathbf{\Lambda}(z) = \begin{bmatrix} z^{-K}\mathbf{I}_r \\ \mathbf{0} \end{bmatrix} \tag{5.10}$$

In view of Theorem 5.2, this gives a second proof that if $\mathbf{g}(K)$ has full rank the FIR inverse is anticausal (since $\ell_i = K \geq 0$ for all i).

We conclude this section by making a related observation. Given any FIR system $\mathbf{G}(z) = \sum_{n=0}^{K} \mathbf{g}(n)z^{-n}$, we can always obtain a direct form implementation with Kr delays (e.g., Fig. 13.9-1 in [8]) so that the degree is at most Kr. When $\mathbf{g}(K)$ has rank r it follows that the degree is precisely Kr (see Example 13.3.2 in [8]). Thus whenever the highest coefficient $\mathbf{g}(K)$ has full rank r the system has McMillan degree Kr, whether the inverse is FIR or not.

6. CONCLUDING REMARKS

The properties of perfect reconstruction filter banks can be conveniently classified according to the nature of the inverse of the polyphase matrix $\mathbf{E}(z)$. The main aim of this paper in this context has been to place in evidence the system theoretic properties of transfer matrices with certain types of inverses. In particular, cases where the inverses are causal, anticausal, and FIR, were considered detail. In [1], we will find further applications of some of these results for the parameterization and factorization of a subclass of causal FIR systems with anticausal FIR inverses (*cafacafi* systems). As noted in Sec. 1.1 such systems are of interest because they can be used to characterize essentially all FIR PR filter banks.

Appendix A. Minimality of anticausal inverse

Assuming that $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ is minimal (i.e., passes the PBH test for reachability and observability [31],[8]) we will verify that $(\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \widehat{\mathbf{C}}, \widehat{\mathbf{D}})$ defined in (3.3) also passes the PBH test. This can be done by contradiction. Assume, for instance, that $(\widehat{\mathbf{C}}, \widehat{\mathbf{A}})$ is not observable. Then there exists $\mathbf{v} \neq \mathbf{0}$ such that $\widehat{\mathbf{A}}\mathbf{v} = \lambda\mathbf{v}$ and $\widehat{\mathbf{C}}\mathbf{v} = \mathbf{0}$. This means

$$\begin{bmatrix} \widehat{\mathbf{A}} & \widehat{\mathbf{B}} \\ \widehat{\mathbf{C}} & \widehat{\mathbf{D}} \end{bmatrix} \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix} = \lambda \begin{bmatrix} \mathbf{v} \\ \mathbf{0} \end{bmatrix}. \tag{A.1}$$

The matrix on the left, which is $\mathcal{R}^{-1}$, is nonsingular, so that $\lambda \neq 0$. Premultiplying both sides of (A.1) by $\mathcal{R}$ and simplifying we obtain $\mathbf{A}\mathbf{v} = \mathbf{v}/\lambda$ and $\mathbf{C}\mathbf{v} = \mathbf{0}$, contradicting the assumed observability of $(\mathbf{C}, \mathbf{A})$.

Appendix B. Generalization of Theorem 5.3

Theorem B.1. *McMillan degree of a system with anticausal inverse.* Let $\mathbf{G}(z)$ be $p \times r$ causal FIR, with an FIR left-inverse. Let N be the degree of a highest-degree $r \times r$ minor. Then N is the McMillan degree of $\mathbf{G}(z)$ if and only if there exists an anticausal FIR inverse. ◊

Note. The above minor need not have the form z^{-n_i}. Example: $\mathbf{G}(z) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1+z^{-1} & 1 \end{bmatrix}$.

Proof. From part 1 of Theorem 5.2 we know that the elements of the Smith-McMillan form are $z^{-\ell_i}$. Thus

$$\text{McMillan degree of } \mathbf{G}(z) = \sum_{\ell_i \geq 0} \ell_i. \tag{B.1}$$

Now recall how the Smith-McMillan form is derived. We first define $\mathbf{G}_1(z)$, a polynomial in z, by writing $\mathbf{G}(z) = z^{-L}\mathbf{G}_1(z)$. Here L is a sufficiently large positive integer. The Smith form of $\mathbf{G}_1(z)$ is a polynomial in z. Since $\mathbf{G}_1(z)$ has an FIR inverse, the diagonal elements of the Smith form of $\mathbf{G}_1(z)$ have the form z^{n_i}, $n_i \geq 0$. (Similar to Theorem 4.1, part 1). So the Smith-McMillan form of $\mathbf{G}(z)$ has diagonal elements z^{-L+n_i}, so that

$$\ell_i = L - n_i. \tag{B.2}$$

From the construction of the Smith form of $\mathbf{G}_1(z)$ we know $z^{n_i} = \Delta_{i+1}(z)/\Delta_i(z)$ where $\Delta_i(z)$ is the greatest common divisor (gcd) of all the $i \times i$ minors of $\mathbf{G}_1(z)$, and $\Delta_0 = 1$ [30]. From this, $\Delta_r(z) = z^{\left(\sum_i n_i\right)}$. This means that all $r \times r$ minors of $\mathbf{G}_1(z)$ are of the form

$$z^{\left(\sum_i n_i\right)} \times [a(0) + a(1)z + a(2)z^2 + \ldots]. \tag{B.3}$$

Since the gcd of all the $r \times r$ minors is $z^{\left(\sum_i n_i\right)}$, at least one of the $r \times r$ minors is such that $a(0) \neq 0$. The $r \times r$ minors of $\mathbf{G}(z)$ therefore have the form

$$z^{\left(-Lr+\sum_i n_i\right)} \times [a(0) + a(1)z + a(2)z^2 + \ldots], \tag{B.4}$$

with at least one of them satisfying $a(0) \neq 0$. Since $\mathbf{G}(z)$ is causal FIR, Eqn. (B.4) is a polynomial in z^{-1} (i.e., the positive powers of z eventually cancel). The largest possible degree of (B.4) therefore comes from those minors with $a(0) \neq 0$, and is equal to $Lr - \sum_{i=0}^{r-1} n_i = \sum_{i=0}^{r-1}(L - n_i) = \sum_{i=0}^{r-1} \ell_i$. Thus the degree of the largest-degree minor of $\mathbf{G}(z)$ is given by

$$\sum_{i=0}^{r-1} \ell_i \leq \sum_{\ell_i \geq 0} \ell_i = \text{McMillan degree of } \mathbf{G}(z) \qquad \text{[from (B.1)]} \tag{B.5}$$

Equality holds if and only if all $\ell_i \geq 0$, that is, if and only if $\mathbf{G}(z)$ has an anticausal inverse (by part 2 of Theorem 5.2). $\nabla\nabla\nabla$

Acknowledgement

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List of figures

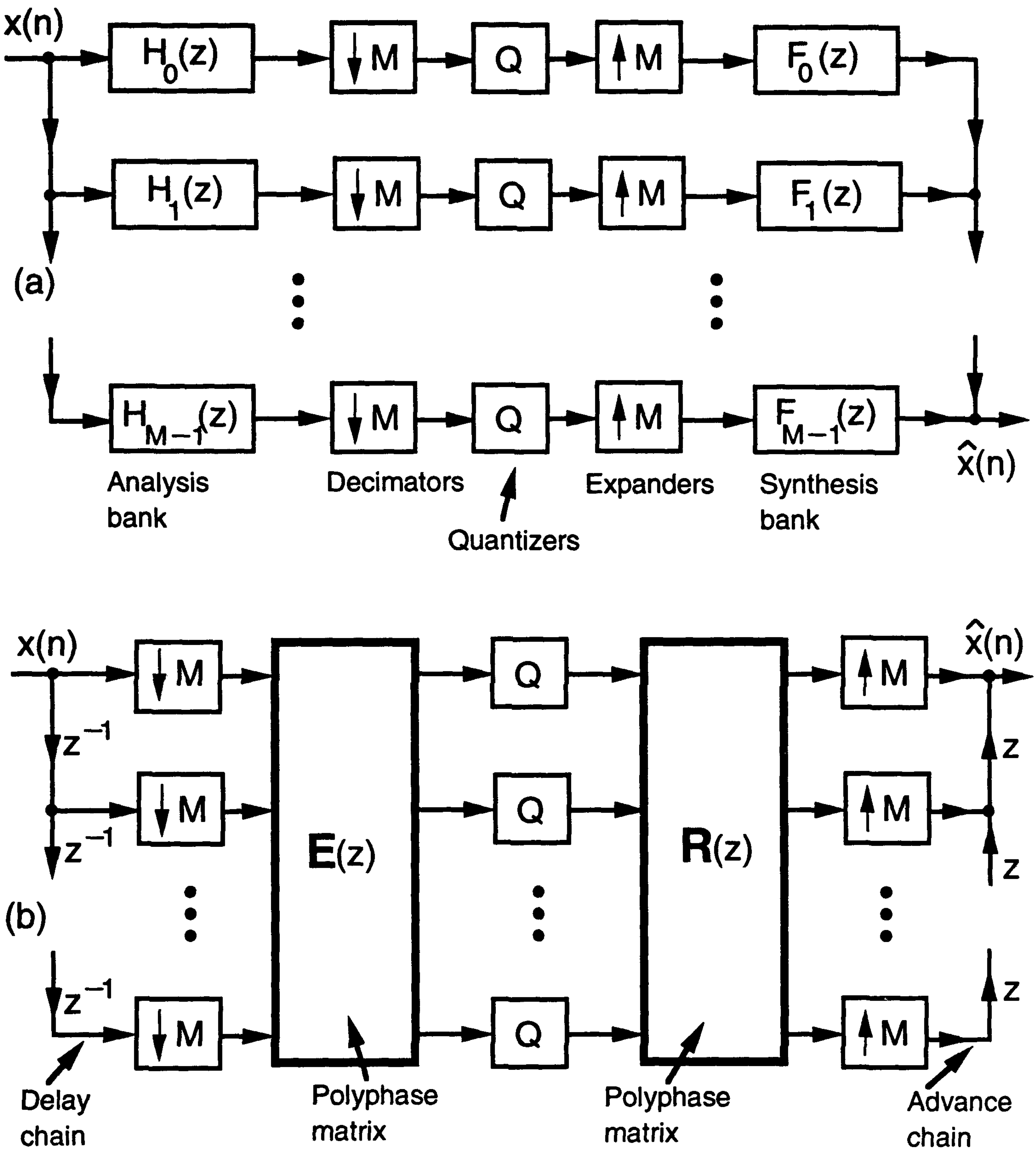

Fig. 1.1 (a) The maximally decimated filter bank, and (b) the polyphase representation.

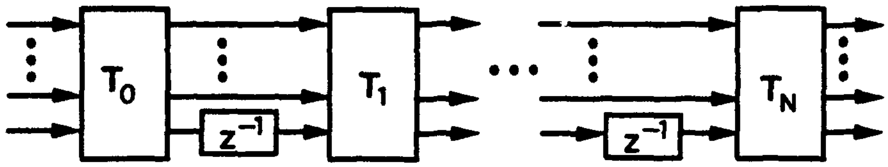

Fig. 1.2. A cascaded structure representing an FIR system with (anticausal) FIR inverse.

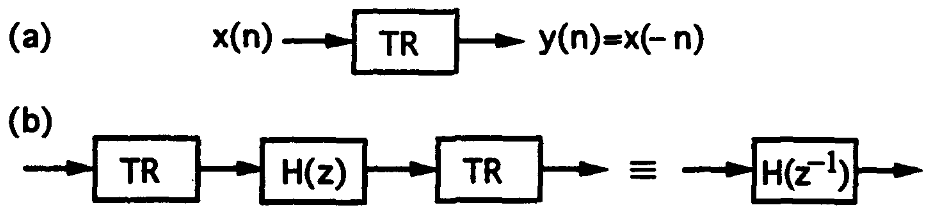

Fig. 1.3 (a) The ideal time-reversal operator, and (b) transformation of an LTI system into another LTI system using time-reversal operators.

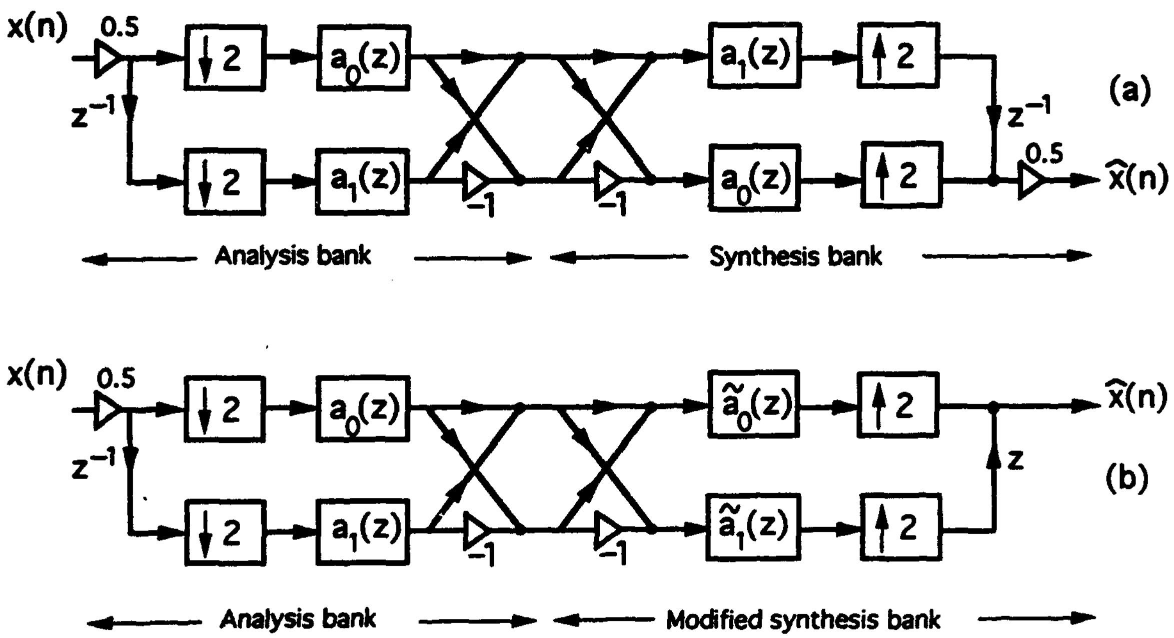

Fig. 2.1 (a) A power-symmetric analysis/synthesis system, and (b) modified version satisfying perfect reconstruction.

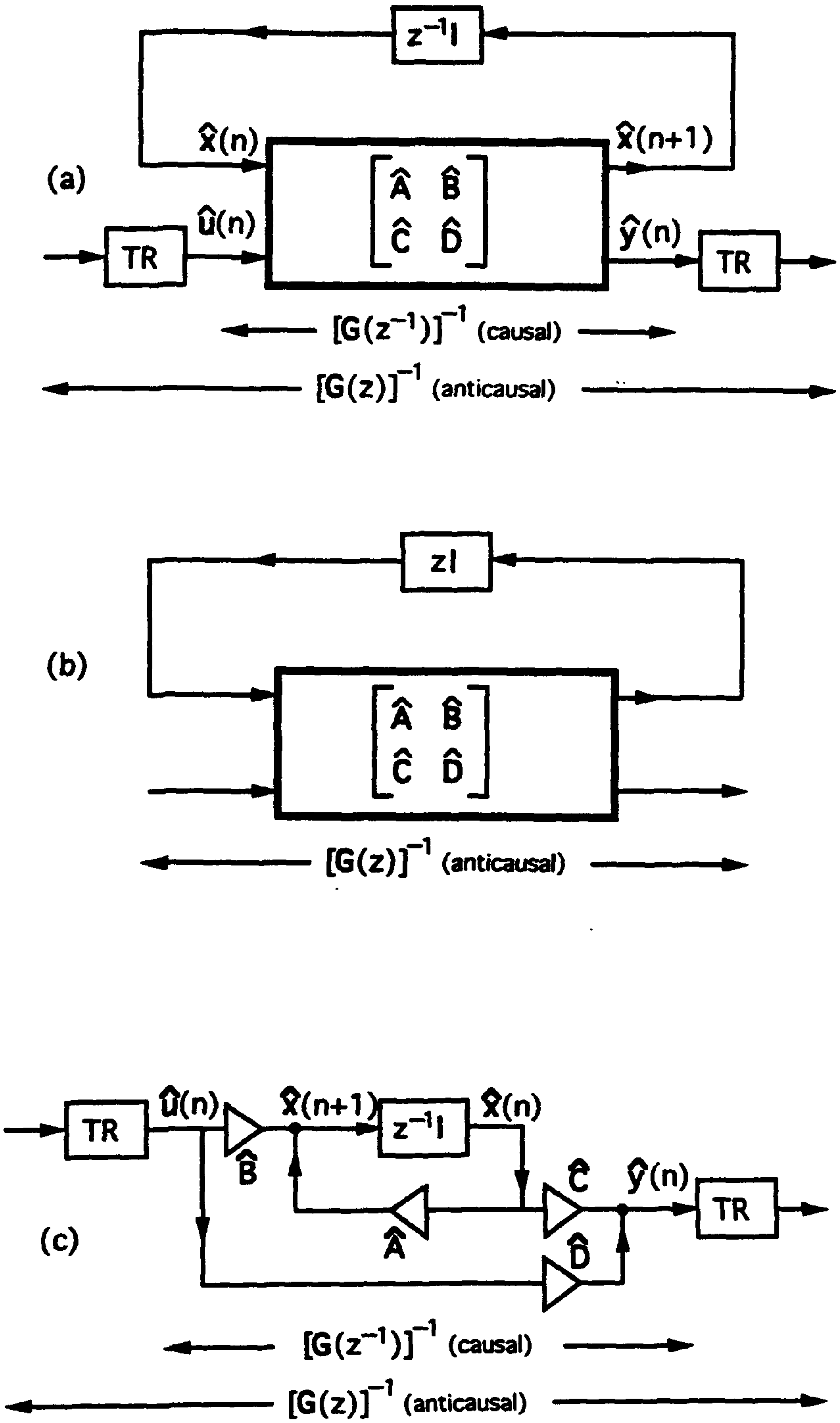

Fig. 3.1 (a) Implementation of an anticausal inverse system by use of inverted realization matrix, (b) simplified schematic, and (c) internal details of the scheme in (a).

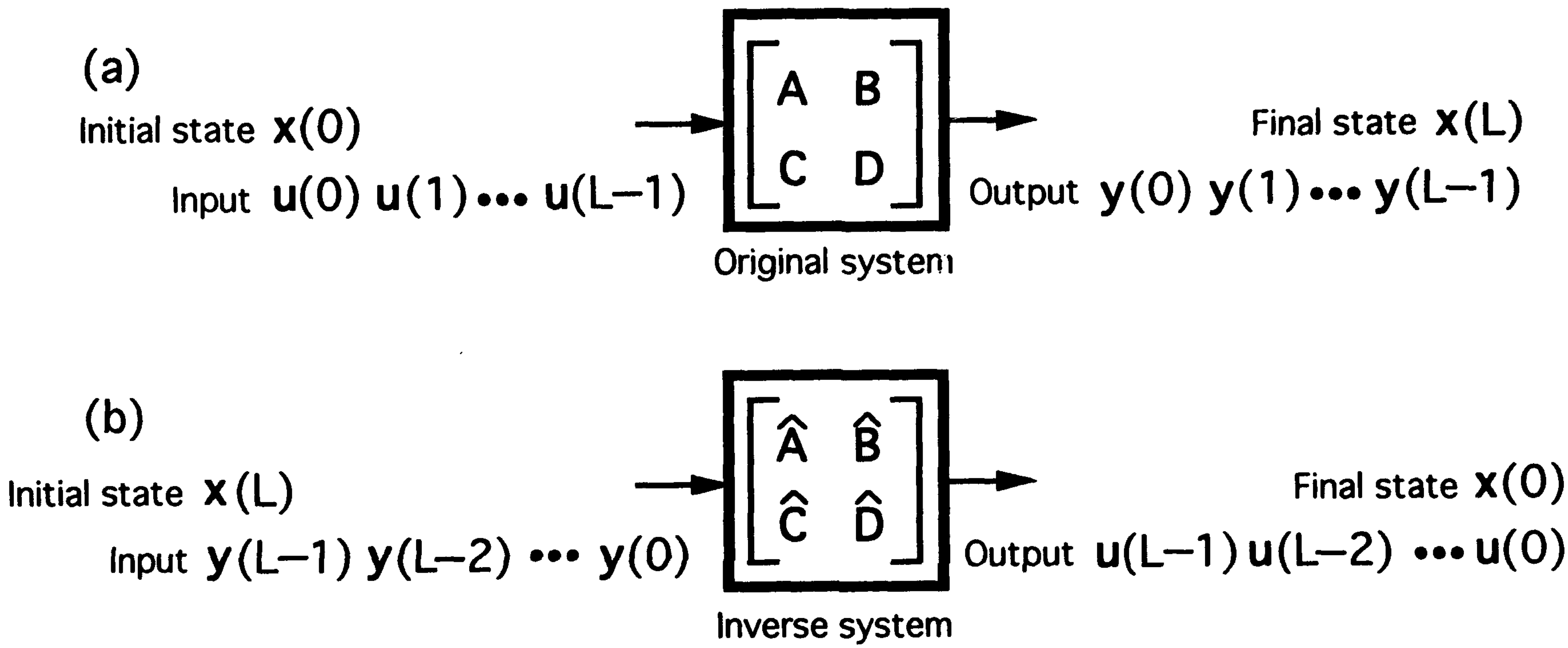

Fig. 3.2. (a) The original system and (b) the inverse system with proper initialization and time reversal.

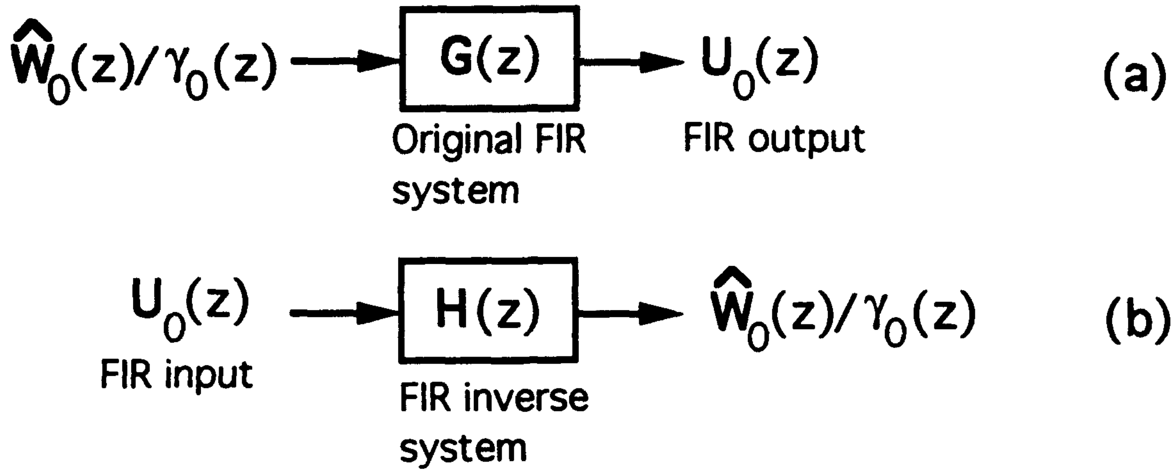

Fig. 4.1. (a) An FIR system with specific choice of input, and (b) the inverse system.

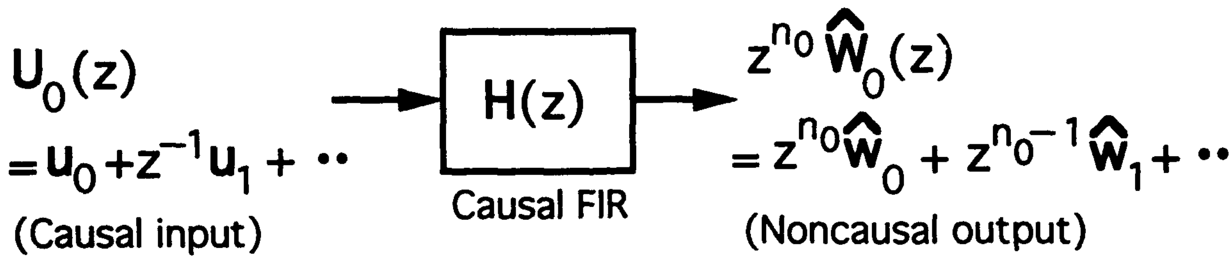

Fig. 4.2. Pertaining to the proof of Theorem 4.1.

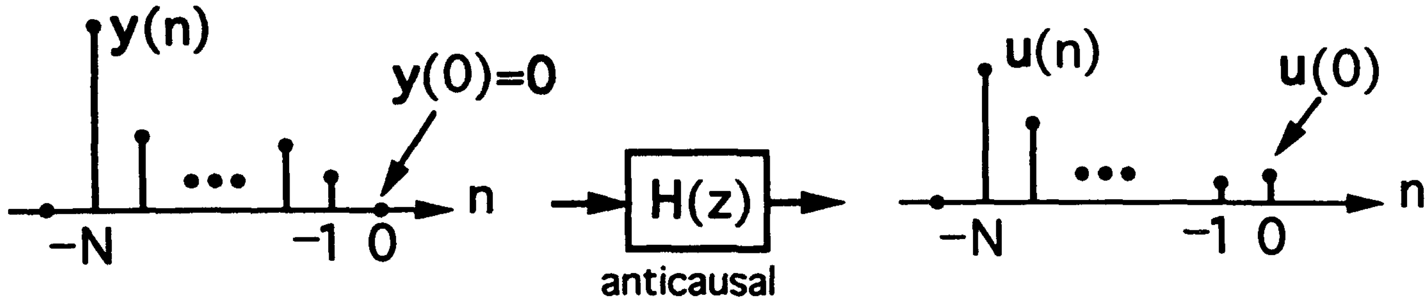

Fig. 5.1. Pertaining to the proof of Theorem 5.1. The above input-output pattern is inconsistent with anticausality.

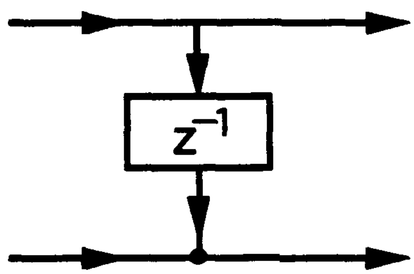

Fig. 5.2. A minimal realization of the system in Example 5.1.

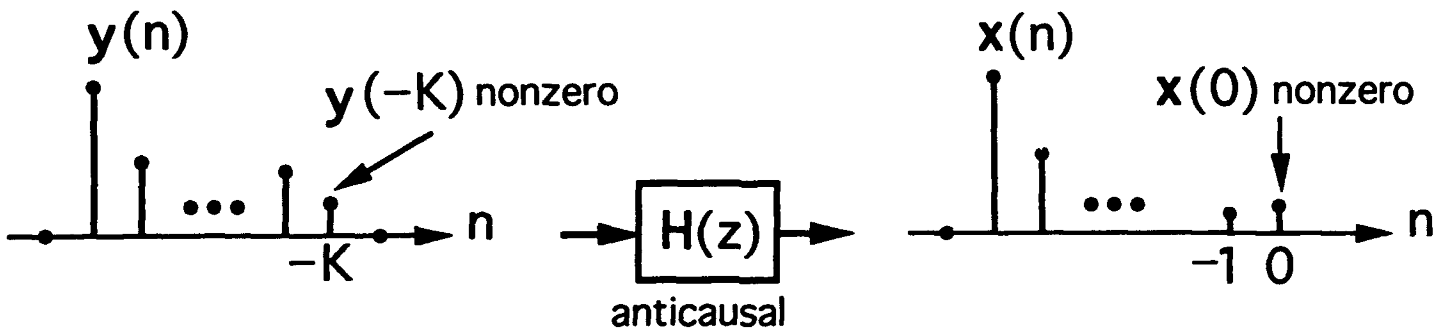

Fig. 5.3. Pertaining to the proof of Theorem 5.2. Anticausality is violated if K is positive.

ROLE OF ANTICAUSAL INVERSES IN MULTIRATE FILTER-BANKS — PART II: THE FIR CASE, FACTORIZATIONS, AND BIORTHONORMAL LAPPED TRANSFORMS

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Abstract. In a companion paper [1], we studied the system-theoretic properties of discrete time transfer matrices in the context of inversion, and classified them according to the types of inverses they had. In particular, we outlined the role of *CAusal Fir matrices with AntiCAusal Fir inverses* (abbreviated *cafacafi*) in the characterization of FIR perfect reconstruction (PR) filter banks. Essentially all FIR PR filter banks can be characterized by causal FIR polyphase matrices having anticausal FIR inverses. In this paper we introduce the most general degree-one *cafacafi* building block, and consider the problem of factorizing *cafacafi* systems into these building blocks. Factrorizability conditions are developed. A special class of *cafacafi* systems called the *biorthonormal lapped transform* (*BOLT*) is developed, and shown to be factorizable. This is a generalization of the well-known lapped orthogonal transform (*LOT*). Examples of unfactorizable *cafacafi* systems are also demonstrated. Finally it is shown that any causal FIR matrix with FIR inverse can be written as a product of a factorizable *cafacafi* system and a unimodular matrix.

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1. INTRODUCTION

In a companion paper [1], we studied the system-theoretic properties of discrete time transfer matrices in the context of inversion, and classified them according to the types of inverses they had. In particular, we outlined the role of *CAusal Fir matrices with AntiCAusal Fir inverses* (abbreviated *cafacafi*) in the characterization of FIR perfect reconstruction filter banks.

Briefly, Fig. 1.1(a) represents a maximally decimated filter bank with identical decimation ratios in all the channels. This can be redrawn in polyphase form as in Fig. 1.1(b). The system has the perfect reconstruction property (i.e., $\widehat{x}(n) = x(n)$ in absence of subband quantizers) if and only if $\mathbf{R}(z) = \mathbf{E}^{-1}(z)$. See [1] for detailed references on this topic. An FIR filter bank is one where $\mathbf{E}(z)$ and $\mathbf{R}(z)$ are FIR. In [1] we argued that in the FIR case, if we study the *cafacafi* class of matrices $\mathbf{E}(z)$, it is sufficient to characterize practically all FIR PR filter banks.

In contrast, the family of causal FIR transfer matrices with causal FIR inverses (i.e., unimodular matrices in z^{-1}) are not very useful in characterizing the class of all FIR PR filter banks. First, restricting the polyphase matrix to be unimodular results in a loss of generality; given a causal FIR system with arbitrary FIR inverse, we cannot in general multiply it with a delay z^{-I} to obtain a causal FIR system with a causal FIR inverse. Furthermore, as we will see at the end of Sec. 2.1, unimodular matrices cannot in general be factorized into degree-one unimodular building blocks.† For these reasons we will not pursue the possibility of characterizing FIR PR systems in terms of unimodular matrices alone. The class of *cafacafi* systems are more useful than unimodular sytems for this purpose.

In this paper we will use the results of [1] to obtain certain fundamental FIR building blocks with FIR inverses. These building blocks can be considered to be the biorthonormal versions of the orthonormal (paraunitary) systems reported earlier [3]–[5]. We will consider the factorization of *cafacafi* systems using these building blocks and develop some results in this direction. For convenience we state from [1] two of the system-theoretic results which play a crucial role in this paper:

1. An $M \times M$ causal LTI system $\mathbf{G}(z)$ has an anticausal inverse if and only if the *realization matrix*

$$\mathcal{R} \triangleq \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \tag{1.1}$$

of any minimal realization of $\mathbf{G}(z)$ is nonsingular (Theorem 5.1 of [1]). Whether the anticausal inverse is FIR or not is not addressed by this result.

† Even though it is well-known [2] that unimodular matrices can be expressed as products of three kinds of elementary matrices, that would not be a useful parameterization for filter bank design.

2. Let $\mathbf{G}(z)$ be an $M \times M$ causal FIR system with FIR inverse. Then the inverse is anticausal FIR if and only if $[\det \mathbf{G}(z)] = cz^{-N}$ where $N =$ McMillan degree of $\mathbf{G}(z)$ (Theorem 5.3 of [1]).

Paper outline

1. In Sec. 2 we present a degree-one building block for *cafacafi* systems, and derive conditions under which arbitrary *cafacafi* systems can be factorized into these building blocks. Even though the building block is the most general degree-one *cafacafi* system (as we show later in Sec. 4) we will see in Sec. 6 that it cannot be used to factorize arbitrary *cafacafi* systems.
2. In Sec. 3 we restate the factorizability conditions in terms of state space parameters.
3. Using this we show in Sec. 4 that a subclass of matrices called the biorthonormal lapped transforms (*BOLT*, a generalization of the lapped orthogonal transform *LOT* [6]–[8]), can always be factorized into degree one *cafacafi* building blocks.
4. In Sec. 5 we study FIR transfer matrices of the form $\mathbf{I}-\mathcal{U}\mathcal{V}^{\dagger}+z^{-1}\mathcal{U}\mathcal{V}^{\dagger}$ and show that many properties of the inverse can be deduced from the eigenvalues of $\mathcal{V}^{\dagger}\mathcal{U}$ (Theorem 5.2). We use this to find necessary and sufficient conditions for any first order FIR matrix to be a *BOLT*. In particular, we impose conditions on the degree-one factors derived in Sec. 2, guaranteeing the *BOLT* property structurally.
5. In Sec. 6 we derive examples of *cafacafi* systems that cannot be factorized into degree one building blocks, and introduce degree two building blocks. It is also shown that there exist *cafacafi* systems which cannot be factorized using any combination of these building blocks.
6. However, in Sec. 7 we show that any causal FIR matrix with FIR inverse can be written as a product of a factorizable *cafacafi* system and a unimodular matrix.

All notations and acronyms will be exactly as in [1].

2. SYNTHESIS USING DEGREE-ONE BUILDING BLOCKS

In this section we introduce the general degree-one causal FIR building block of the form

$$\mathbf{V}(z) = \mathbf{I} - \mathbf{u}\mathbf{v}^{\dagger} + z^{-1}\mathbf{u}\mathbf{v}^{\dagger} \tag{2.1}$$

where $\mathbf{u}$ and $\mathbf{v}$ are $M \times 1$ vectors, and study its properties. In particular, its role in the synthesis of FIR causal systems with anticausal FIR inverses will be studied. Because of the appearance of the outer product $\mathbf{u}\mathbf{v}^{\dagger}$, the building block is said to be *diadic-based*. Fig. 2.1 shows a structure for this system. Note that $\mathbf{V}(1) = \mathbf{I}$.

2.1. Properties of the degree-one building block

Theorem 2.1. Consider the $M \times M$ system $\mathbf{V}(z) = \mathbf{I} - \mathbf{u}\mathbf{v}^\dagger + z^{-1}\mathbf{u}\mathbf{v}^\dagger$ where $\mathbf{u}$ and $\mathbf{v}$ are $M \times 1$ vectors (so that the degree = 1 unless $\mathbf{u}$ or $\mathbf{v}$ is zero). Then the following are true.

1. $[\det \mathbf{V}(z)] = 1 + \mathbf{v}^\dagger\mathbf{u}(z^{-1} - 1)$.
2. Let $\mathbf{u}^\dagger\mathbf{v} = 1$, so that $[\det \mathbf{V}(z)] = z^{-1}$. In this case, $\mathbf{V}^{-1}(z) = \mathbf{V}(z^{-1}) = \mathbf{I} - \mathbf{u}\mathbf{v}^\dagger + z\mathbf{u}\mathbf{v}^\dagger$. That is, the inverse is anticausal FIR. If $\mathbf{u} = \mathbf{v}$, then $\mathbf{V}(z)$ becomes the paraunitary building block known before [5].
3. Let $\mathbf{u}^\dagger\mathbf{v} = 0$, so that $[\det \mathbf{V}(z)] = 1$ (i.e., $\mathbf{V}(z)$ is unimodular in z^{-1}). In this case $\mathbf{V}^{-1}(z) = \mathbf{I} + \mathbf{u}\mathbf{v}^\dagger - z^{-1}\mathbf{u}\mathbf{v}^\dagger$ which is causal FIR. $\diamond$

Proof. Let $\mathbf{x}_i$, $0 \le i \le M-2$ be vectors orthogonal to $\mathbf{v}$. Then $\mathbf{V}(z)\mathbf{x}_i = \mathbf{x}_i$ so that there are $M-1$ eigenvectors with eigenvalue unity. Next, by substitution we see that $\mathbf{V}(z)\mathbf{u} = \big(1 + \mathbf{v}^\dagger\mathbf{u}(z^{-1}-1)\big)\mathbf{u}$ so that $\big(1 + \mathbf{v}^\dagger\mathbf{u}(z^{-1}-1)\big)$ is an eigenvalue. When $\mathbf{v}^\dagger\mathbf{u} \ne 0$, $\mathbf{u}$ is not in the span of $\{\mathbf{x}_i\}$. So we have found M independent eigenvectors including $\mathbf{u}$, and all but one have eigenvalue equal to unity. Thus

$$\det \mathbf{V}(z) = 1 + \mathbf{v}^\dagger\mathbf{u}\big(z^{-1} - 1\big). \tag{2.2}$$

When $\mathbf{v}^\dagger\mathbf{u} = 0$ it can be shown that there are no eigenvectors of $\mathbf{V}(z)$ other than the $\mathbf{x}_i$ (or their linear combinations). For this note that $\mathbf{V}(z)\mathbf{w} = \mathbf{w} + (\mathbf{v}^\dagger\mathbf{w})(z^{-1}-1)\mathbf{u}$ for any $\mathbf{w}$. If $\mathbf{w}$ is an eigenvector, then either (i) $\mathbf{w}$ is aligned to $\mathbf{u}$ or (ii) $\mathbf{v}^\dagger\mathbf{w} = 0$. Since $\mathbf{v}^\dagger\mathbf{u} = 0$, condition (i) implies $\mathbf{v}^\dagger\mathbf{w} = 0$ which is condition (ii) again. The condition $\mathbf{v}^\dagger\mathbf{w} = 0$ means, of course, that $\mathbf{w}$ is a linear combination of $\mathbf{x}_i$'s. So all the eigenvectors are in the span of $\mathbf{x}_i$'s, and the common eigenvalue is unity. Thus $[\det \mathbf{V}(z)] = 1$. That is, (2.2) holds even with $\mathbf{v}^\dagger\mathbf{u} = 0$.

The stated forms of the inverses in parts 2 and 3 can be verified by direct multiplication of $\mathbf{V}(z)$ with the claimed inverse and using $\mathbf{u}^\dagger\mathbf{v} = 0$ or 1 as the case may be. $\nabla\nabla\nabla$

Comments

1. For $\mathbf{u}^\dagger\mathbf{v} = 1$, the following identity is easily verified:

$$\mathbf{I} - \mathbf{u}\mathbf{v}^\dagger + z^{-K}\mathbf{u}\mathbf{v}^\dagger = \prod_{K \text{ times}} \Big(\mathbf{I} - \mathbf{u}\mathbf{v}^\dagger + z^{-1}\mathbf{u}\mathbf{v}^\dagger\Big).$$

2. *Smith-McMillan forms.* (Reviewed in Sec. 4.1 of [1]). Since $\mathbf{u}^\dagger\mathbf{v} = 1$ implies that $\mathbf{V}(z)$ has an anticausal FIR inverse, so by Theorem 5.2 in [1] the Smith-McMillan form of $\mathbf{V}(z)$ is $\begin{bmatrix} z^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$. On the other hand, $\mathbf{u}^\dagger\mathbf{v} = 0$ implies that $\mathbf{V}(z)$ is unimodular in z^{-1}, and the Smith-McMillan form of $\mathbf{V}(z)$ is the identity matrix modified as follows: the first diagonal element is replaced with z^{-1} and the last diagonal element replaced with z.
3. Let $\mathbf{u}\mathbf{v}^\dagger \ne \mathbf{0}$ to avoid trivialities. Then we can show the following: $\mathbf{V}(z)$ has an (i) anticausal inverse if and only if $\mathbf{v}^\dagger\mathbf{u} \ne 0$, (ii) FIR inverse if and only if $\mathbf{v}^\dagger\mathbf{u} = 0$ or 1. These will follow as special cases

of a more general result (Theorem 5.2). Thus the inverse is anticausal FIR if and only if $\mathbf{v}^\dagger\mathbf{u} = 1$ and causal FIR if and only if $\mathbf{v}^\dagger\mathbf{u} = 0$.

A related Unimodular system

Consider the $M \times M$ causal FIR system $\mathbf{V}(z) = \mathbf{I} + z^{-1}\mathbf{u}\mathbf{v}^\dagger$, where $\mathbf{u}$ and $\mathbf{v}$ are $M \times 1$ vectors. If $\mathbf{v}^\dagger\mathbf{u} = 0$ it can be verified that the inverse is $\mathbf{I} - z^{-1}\mathbf{u}\mathbf{v}^\dagger$, so that $\mathbf{V}(z)$ is unimodular. A stronger result is the following.

Lemma 2.1. The system $\mathbf{V}(z) = \mathbf{I} + z^{-1}\mathbf{u}\mathbf{v}^\dagger$ has IIR inverse if $\mathbf{v}^\dagger\mathbf{u} \neq 0$, and causal FIR inverse when $\mathbf{v}^\dagger\mathbf{u} = 0$. So $\mathbf{V}(z)$ is unimodular if and only if $\mathbf{v}^\dagger\mathbf{u} = 0$. $\diamond$

Proof. Let $\mathbf{x}_i$, $1 \leq i \leq M-1$ be a set of independent vectors orthogonal to $\mathbf{v}$. Then $\mathbf{V}(z)\mathbf{x}_i = \mathbf{x}_i$. On the other hand $\mathbf{V}(z)\mathbf{u} = (1 + z^{-1}\mathbf{v}^\dagger\mathbf{u})\mathbf{u}$. So

$$\mathbf{V}(z)\begin{bmatrix}\mathbf{u} & \mathbf{P}\end{bmatrix} = \begin{bmatrix}(1 + z^{-1}\mathbf{v}^\dagger\mathbf{u})\mathbf{u} & \mathbf{P}\end{bmatrix},$$

where $\mathbf{P}$ is $M \times (M-1)$ with columns equal to $\mathbf{x}_i$. If $\mathbf{v}^\dagger\mathbf{u} \neq 0$, the vector $\mathbf{u}$ is not in the span of $\mathbf{x}_i$. So $\begin{bmatrix}\mathbf{u} & \mathbf{P}\end{bmatrix}$ is nonsingular and we get $[\det \mathbf{V}(z)] = (1 + z^{-1}\mathbf{v}^\dagger\mathbf{u})$. This is not a delay since $\mathbf{v}^\dagger\mathbf{u} \neq 0$. So the inverse of $\mathbf{V}(z)$ is IIR. $\nabla\nabla\nabla$

More generally, let $\mathbf{G}(z)$ be any degree-one unimodular matrix in z^{-1}. Since $\mathbf{G}(\infty)$ is nonsingular, we can always write $\mathbf{G}(z) = (\mathbf{I} + z^{-1}\mathbf{u}\mathbf{v}^\dagger)\mathbf{D}$ where $\mathbf{v}^\dagger\mathbf{u} = 0$ and $\mathbf{D} = \mathbf{G}(\infty)$.

A degree-two unfactorizable unimodular system. We now show that the unimodular system

$$\mathbf{G}(z) = \begin{bmatrix} 1 & 0 \\ z^{-2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + z^{-2}\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \tag{2.3}$$

cannot be factorized into degree-one causal unimodular systems. Suppose we could, then

$$\mathbf{G}(z) = \left(\mathbf{D}_0 + z^{-1}\mathbf{u}_0\mathbf{v}_0^\dagger\right)\left(\mathbf{D}_1 + z^{-1}\mathbf{u}_1\mathbf{v}_1^\dagger\right), \tag{2.4}$$

where $\mathbf{D}_0$ and $\mathbf{D}_1$ are nonsingular and must be such that $\mathbf{D}_0\mathbf{D}_1 = \mathbf{I}$. We can always rearrange this to be of the form $\mathbf{G}(z) = (\mathbf{I} + z^{-1}\mathbf{u}_0\mathbf{v}_0^\dagger)(\mathbf{I} + z^{-1}\mathbf{u}_1\mathbf{v}_1^\dagger)$ by redefining the vectors $\mathbf{u}_i$ and $\mathbf{v}_i$. Comparison of coefficients of z^{-1} in (2.3) and the product $(\mathbf{I} + z^{-1}\mathbf{u}_0\mathbf{v}_0^\dagger)(\mathbf{I} + z^{-1}\mathbf{u}_1\mathbf{v}_1^\dagger)$ shows that we need $\mathbf{u}_0\mathbf{v}_0^\dagger + \mathbf{u}_1\mathbf{v}_1^\dagger = \mathbf{0}$ so that $\mathbf{u}_1 = c\mathbf{u}_0$ for some scalar c. This implies $\mathbf{v}_0^\dagger\mathbf{u}_1 = 0$, since $\mathbf{v}_0^\dagger\mathbf{u}_0 = 0$ for unimodularity of $(\mathbf{I} + z^{-1}\mathbf{u}_0\mathbf{v}_0^\dagger)$. Thus the coefficient of z^{-2} in (2.4) is $\mathbf{u}_0(\mathbf{v}_0^\dagger\mathbf{u}_1)\mathbf{v}_1^\dagger = \mathbf{0}$, and the product (2.4) can never be equal to (2.3).

2.2. Degree reduction using degree-one building blocks

We are given an $M \times M$ causal FIR matrix $\mathbf{G}_m(z)$ with anticausal FIR inverse $\mathbf{H}_m(z)$:

$$\mathbf{G}_m(z) = \sum_{n=0}^{K} z^{-n}\mathbf{g}_m(n), \quad \mathbf{H}_m(z) = \sum_{n=0}^{L} z^{n}\mathbf{h}_m(n). \tag{2.5}$$

Assume $K, L > 0$ and $\mathbf{g}_m(K) \neq \mathbf{0}$ and $\mathbf{h}_m(L) \neq \mathbf{0}$ to avoid trivialities. Then K is the order of $\mathbf{G}_m(z)$ and L is the order of $\mathbf{H}_m(z)$. If the McMillan degree of $\mathbf{G}_m(z)$ is m, then

$$[\det \mathbf{G}_m(z)] = cz^{-m}$$

(Theorem 5.3, [1]). Suppose we wish to express it as

$$\mathbf{G}_m(z) = \mathbf{V}_m(z)\mathbf{G}_{m-1}(z) \tag{2.6}$$

where $\mathbf{V}_m(z)$ is a degree-one causal FIR system with anticausal FIR inverse:

$$\mathbf{V}_m(z) = \mathbf{I} - \mathbf{u}\mathbf{v}^\dagger + z^{-1}\mathbf{u}\mathbf{v}^\dagger, \quad \mathbf{v}^\dagger\mathbf{u} = 1. \tag{2.7}$$

See Fig. 2.2(a). From Theorem 2.1 we have $[\det \mathbf{V}_m(z)] = z^{-1}$ so Eqn. (2.6) implies $[\det \mathbf{G}_{m-1}(z)] = cz^{-(m-1)}$. So we know that $\mathbf{G}_{m-1}(z)$ has McMillan degree $(m-1)$ as long as it is also causal FIR with anticausal FIR inverse (Theorem 5.3, [1]). If we can do this successfully m times, then the final remainder $\mathbf{G}_0(z)$ is *cafacafi* with constant determinant so that it is just a nonsingular constant (Theorem 5.3, [1]). This would give the cascaded structure of Fig. 2.2(b).

It only remains to explore the conditions under which we can successfully ensure that $\mathbf{G}_{m-1}(z)$ is *cafacafi*. Since $\mathbf{V}_m^{-1}(z) = \mathbf{V}_m(z^{-1})$ (Theorem 2.1), we can write

$$\mathbf{G}_{m-1}(z) = \Big(\mathbf{I} - \mathbf{u}\mathbf{v}^\dagger + z\mathbf{u}\mathbf{v}^\dagger\Big)\Big(\mathbf{g}_m(0) + z^{-1}\mathbf{g}_m(1) + \ldots + z^{-K}\mathbf{g}_m(K)\Big). \tag{2.8}$$

Causality of the remainder $\mathbf{G}_{m-1}(z)$ requires $\mathbf{v}^\dagger\mathbf{g}_m(0) = \mathbf{0}$. Next consider

$$\mathbf{G}_{m-1}^{-1}(z) = \Big(\mathbf{h}_m(0) + z\mathbf{h}_m(1) + \ldots + z^L\mathbf{h}_m(L)\Big)\Big(\mathbf{I} - \mathbf{u}\mathbf{v}^\dagger + z^{-1}\mathbf{u}\mathbf{v}^\dagger\Big). \tag{2.9}$$

Anticausality of this quantity requires $\mathbf{h}_m(0)\mathbf{u} = \mathbf{0}$. Summarizing, the degree-reduction procedure will succeed if and only if there exist vectors $\mathbf{u}$ and $\mathbf{v}$ such that

$$\mathbf{v}^\dagger\mathbf{g}_m(0) = \mathbf{0}, \quad \mathbf{h}_m(0)\mathbf{u} = \mathbf{0}, \quad \mathbf{v}^\dagger\mathbf{u} = 1. \tag{2.10}$$

We know that $\mathbf{g}_m(0)$ and $\mathbf{h}_m(0)$ are singular (Sec. 5.3, [1]) and therefore there exist nonnull vectors $\mathbf{v}$ and $\mathbf{u}$ satisfying $\mathbf{v}^\dagger\mathbf{g}_m(0) = \mathbf{0}$ and $\mathbf{h}_m(0)\mathbf{u} = \mathbf{0}$. However there is no guarantee that there will exist $\mathbf{u}$ and $\mathbf{v}$ which are also nonorthogonal (so that they can be scaled to satisfy $\mathbf{v}^\dagger\mathbf{u} = 1$).

In Sec. 6 we will see examples of *cafacafi* $\mathbf{G}_m(z)$ for which (2.10) cannot be satisfied. In Sec. 4–5 we will present some useful subclasses of *cafacafi* systems for which (2.10) can be satisfied at every step of the degree reduction process. Towards this goal it proves to be convenient to reformulate the condition (2.10) in terms of the state space descriptions $(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ and $(\widehat{\mathbf{A}},\widehat{\mathbf{B}},\widehat{\mathbf{C}},\widehat{\mathbf{D}})$

3. STATE SPACE FORMULATION OF FACTORIZABILITY

In Sec. 3 of [1] we described causal systems having anticausal inverses in terms of minimal state space descriptions. Let $(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ be a minimal realization of $\mathbf{G}_m(z)$. Defining the realization matrix $\mathcal{R}$ and its inverse

$$\mathcal{R}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}, \qquad \begin{bmatrix}\widehat{\mathbf{A}} & \widehat{\mathbf{B}}\\ \widehat{\mathbf{C}} & \widehat{\mathbf{D}}\end{bmatrix}=\begin{bmatrix}\mathbf{A} & \mathbf{B}\\ \mathbf{C} & \mathbf{D}\end{bmatrix}^{-1}=\mathcal{R}^{-1} \tag{3.1}$$

we obtain the minimal realization $(\widehat{\mathbf{A}},\widehat{\mathbf{B}},\widehat{\mathbf{C}},\widehat{\mathbf{D}})$ for the anticausal inverse in the sense defined in Sec. 3.1 of [1]. (We have omitted a subscript m on the matrices $(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ etc., for simplicity). Note that the inverse of $\mathcal{R}$ exists because of the assumed existence of the anticausal inverse (Theorem 5.1 of [1]). We can express

$$\mathbf{G}_m(z)=\mathbf{D}+\sum_{n=1}^{K}z^{-n}\mathbf{C}\mathbf{A}^{n-1}\mathbf{B}, \quad \mathbf{H}_m(z)=\mathbf{G}_m^{-1}(z)=\widehat{\mathbf{D}}+\sum_{n=1}^{L}z^{n}\widehat{\mathbf{C}}\widehat{\mathbf{A}}^{n-1}\widehat{\mathbf{B}} \tag{3.2}$$

In particular, therefore, $\mathbf{D}=\mathbf{g}_m(0)$ and $\widehat{\mathbf{D}}=\mathbf{h}_m(0)$, so the three conditions in (2.10) are equivalent to

$$\mathbf{v}^{\dagger}\mathbf{D}=\mathbf{0}, \quad \widehat{\mathbf{D}}\mathbf{u}=\mathbf{0}, \quad \mathbf{v}^{\dagger}\mathbf{u}=1. \tag{3.3}$$

As stated before, $\mathbf{D}$ and $\widehat{\mathbf{D}}$ are singular, so the only nontrivial issue is to prove the existence of $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{v}^{\dagger}\mathbf{u}=1$.

In all the results to follow, $(\mathbf{A},\mathbf{B},\mathbf{C},\mathbf{D})$ and $(\widehat{\mathbf{A}},\widehat{\mathbf{B}},\widehat{\mathbf{C}},\widehat{\mathbf{D}})$ are minimal realizations of $\mathbf{G}_m(z)$ and $\mathbf{G}_m^{-1}(z)$ respectively, and are related as in (3.1). Note that since $\mathbf{G}_m(z)$ and $\mathbf{G}_m^{-1}(z)$ are FIR, all the eigenvalues of $\mathbf{A}$ and $\widehat{\mathbf{A}}$ are equal to zero. By explicitly writing out the four components of the relation $\mathcal{R}\mathcal{R}^{-1}=\mathbf{I}$, we obtain the four equations

$$\mathbf{A}\widehat{\mathbf{A}}+\mathbf{B}\widehat{\mathbf{C}}=\mathbf{I}, \quad \mathbf{A}\widehat{\mathbf{B}}+\mathbf{B}\widehat{\mathbf{D}}=\mathbf{0}, \quad \mathbf{C}\widehat{\mathbf{A}}+\mathbf{D}\widehat{\mathbf{C}}=\mathbf{0}, \quad \mathbf{C}\widehat{\mathbf{B}}+\mathbf{D}\widehat{\mathbf{D}}=\mathbf{I} \tag{3.4}$$

Similarly, by writing out $\mathcal{R}^{-1}\mathcal{R}=\mathbf{I}$, we get

$$\widehat{\mathbf{A}}\mathbf{A}+\widehat{\mathbf{B}}\mathbf{C}=\mathbf{I}, \quad \widehat{\mathbf{A}}\mathbf{B}+\widehat{\mathbf{B}}\mathbf{D}=\mathbf{0}, \quad \widehat{\mathbf{C}}\mathbf{A}+\widehat{\mathbf{D}}\mathbf{C}=\mathbf{0}, \quad \widehat{\mathbf{C}}\mathbf{B}+\widehat{\mathbf{D}}\mathbf{D}=\mathbf{I} \tag{3.5}$$

We will find these equations useful for future reference.

Theorem 3.1. There exist vectors $\mathbf{u}$ and $\mathbf{v}$ satisfying $\mathbf{v}^{\dagger}\mathbf{D}=\mathbf{0}$, $\widehat{\mathbf{D}}\mathbf{u}=\mathbf{0}$, and $\mathbf{v}^{\dagger}\mathbf{u}=1$ if and only if there exist vectors $\mathbf{t}$ and $\mathbf{s}$ satisfying

$$\mathbf{A}\mathbf{s}=\mathbf{0}, \quad \mathbf{t}^{\dagger}\widehat{\mathbf{A}}=\mathbf{0}, \quad \text{and} \quad \mathbf{t}^{\dagger}\mathbf{s}=1. \tag{3.6}$$

Proof. Suppose $\mathbf{v}^\dagger\mathbf{D} = \mathbf{0}$ for some $\mathbf{v}$. From (3.4) we see that this implies $\mathbf{v}^\dagger\mathbf{C}\widehat{\mathbf{A}} = \mathbf{0}$ and $\mathbf{v}^\dagger\mathbf{C}\widehat{\mathbf{B}} = \mathbf{v}^\dagger$. Defining $\mathbf{t}^\dagger = \mathbf{v}^\dagger\mathbf{C}$ we see that $\mathbf{t}^\dagger\widehat{\mathbf{A}} = \mathbf{0}$. Similarly we can show using (3.4) that if $\widehat{\mathbf{D}}\mathbf{u} = \mathbf{0}$ for some $\mathbf{u}$ then $\mathbf{A}\mathbf{s} = \mathbf{0}$ where $\mathbf{s} = \widehat{\mathbf{B}}\mathbf{u}$. With the quantities $\mathbf{t}$ and $\mathbf{s}$ defined in terms of $\mathbf{v}$ and $\mathbf{u}$ as above, we get

$$\mathbf{t}^\dagger\mathbf{s} = \mathbf{v}^\dagger\mathbf{C}\widehat{\mathbf{B}}\mathbf{u} = \mathbf{v}^\dagger\mathbf{u}$$

using $\mathbf{C}\widehat{\mathbf{B}} = \mathbf{I} - \mathbf{D}\widehat{\mathbf{D}}$ [from (3.4)] and the fact that $\mathbf{v}^\dagger\mathbf{D} = \mathbf{0}$. Summarizing, if there exist $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{v}^\dagger\mathbf{D} = \mathbf{0}$ and $\widehat{\mathbf{D}}\mathbf{u} = \mathbf{0}$ then there exist $\mathbf{t}$ and $\mathbf{s}$ such that $\mathbf{t}^\dagger\widehat{\mathbf{A}} = \mathbf{0}$, $\mathbf{A}\mathbf{s} = \mathbf{0}$, and $\mathbf{t}^\dagger\mathbf{s} = \mathbf{v}^\dagger\mathbf{u}$.

Second, suppose there exist vectors $\mathbf{s}$ and $\mathbf{t}$ such that $\mathbf{A}\mathbf{s} = \mathbf{0}$ and $\mathbf{t}^\dagger\widehat{\mathbf{A}} = \mathbf{0}$. Defining $\mathbf{u} = \mathbf{C}\mathbf{s}$ and $\mathbf{v}^\dagger = \mathbf{t}^\dagger\widehat{\mathbf{B}}$ we can show using (3.5) that $\widehat{\mathbf{D}}\mathbf{u} = \mathbf{0}$ and $\mathbf{v}^\dagger\mathbf{D} = \mathbf{0}$, and furthermore $\mathbf{v}^\dagger\mathbf{u} = \mathbf{t}^\dagger\mathbf{s}$. Combining this with the observation in the preceding paragraph, we can say that there exist vectors $\mathbf{u}$ and $\mathbf{v}$ satisfying $\mathbf{v}^\dagger\mathbf{D} = \mathbf{0}$, $\widehat{\mathbf{D}}\mathbf{u} = \mathbf{0}$, and $\mathbf{v}^\dagger\mathbf{u} = 1$ if and only if there exist vectors $\mathbf{t}$ and $\mathbf{s}$ satisfying $\mathbf{A}\mathbf{s} = \mathbf{0}, \mathbf{t}^\dagger\widehat{\mathbf{A}} = \mathbf{0}$, and $\mathbf{t}^\dagger\mathbf{s} = 1$. $\nabla\nabla\nabla$

In the above theorem we have established a one to one correspondence between the annihilators of the pair $(\mathbf{D}, \widehat{\mathbf{D}})$ and the pair $(\mathbf{A}, \widehat{\mathbf{A}})$. So the degree reduction condition for the *cafacafi* factorization can be reformulated as follows:

Theorem 3.2. The degree reduction step for the causal FIR system $\mathbf{G}_m(z)$ with anticausal FIR inverse $\mathbf{G}_m^{-1}(z)$ will be successful if and only if there exist vectors $\mathbf{t}$ and $\mathbf{s}$ satisfying (3.6), or equivalently vectors $\mathbf{u}$ and $\mathbf{v}$ satisfying (3.3). $\diamondsuit$

A different state-space condition

With $\mathbf{G}_m(z)$ and $\mathbf{H}_m(z)$ expressed as in (2.5), we know that $\mathbf{h}(0)\mathbf{g}(K) = \mathbf{0}$ and $\mathbf{h}(L)\mathbf{g}(0) = \mathbf{0}$ (subscript m on $\mathbf{g}(n)$ and $\mathbf{h}(n)$ omitted for convenience). This shows that we can satisfy (2.10) by taking $\mathbf{v}^\dagger$ to be any row of $\mathbf{h}(L)$ and $\mathbf{u}$ to be any column of $\mathbf{g}(K)$. There will exist such a choice which further satisfies the condition $\mathbf{v}^\dagger\mathbf{u} = 1$ as long as $\mathbf{h}(L)\mathbf{g}(K) \neq \mathbf{0}$. In this connection, the following result is helpful.

Theorem 3.3. Consider the $M \times M$ system $\mathbf{G}_m(z) = \sum_{i=0}^{K} z^{-i}\mathbf{g}(i)$ with anticausal FIR inverse $\mathbf{G}_m^{-1}(z) = \sum_{i=0}^{L} z^{i}\mathbf{h}(i)$. Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ and $(\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \widehat{\mathbf{C}}, \widehat{\mathbf{D}})$ represent their respective minimal realizations related in the usual manner, i.e., as in (3.1). Then $\mathbf{h}(L)\mathbf{g}(K) = \mathbf{0}$ if and only if $\widehat{\mathbf{A}}^{L-1}\mathbf{A}^{K-1} = \mathbf{0}$. $\diamondsuit$

Proof. We know $\mathbf{h}(L) = \widehat{\mathbf{C}}\widehat{\mathbf{A}}^{L-1}\widehat{\mathbf{B}}$ and $\mathbf{g}(K) = \mathbf{C}\mathbf{A}^{K-1}\mathbf{B}$, so that

$$\begin{aligned}\mathbf{h}(L)\mathbf{g}(K) &= \widehat{\mathbf{C}}\widehat{\mathbf{A}}^{L-1}\widehat{\mathbf{B}}\mathbf{C}\mathbf{A}^{K-1}\mathbf{B} = \widehat{\mathbf{C}}\widehat{\mathbf{A}}^{L-1}\mathbf{A}^{K-1}\mathbf{B} - \widehat{\mathbf{C}}\widehat{\mathbf{A}}^{L}\mathbf{A}^{K}\mathbf{B} \quad \text{(using (3.5))}\\ &= \widehat{\mathbf{C}}\widehat{\mathbf{A}}^{L-1}\mathbf{A}^{K-1}\mathbf{B}.\end{aligned} \tag{3.7}$$

The last equality follows because the FIR property and minimality of the realizations imply $\widehat{\mathbf{C}}\widehat{\mathbf{A}}^{L} = \mathbf{0}$ and

$\mathbf{A}^K\mathbf{B} = \mathbf{0}$ (Lemma 13.9.1, [4]). Now consider the product

$$\underbrace{\begin{bmatrix} \widehat{\mathbf{C}} \\ \widehat{\mathbf{C}}\widehat{\mathbf{A}} \\ \vdots \\ \widehat{\mathbf{C}}\widehat{\mathbf{A}}^{m-1} \end{bmatrix}}_{\mathbf{P}} \widehat{\mathbf{A}}^{L-1}\mathbf{A}^{K-1} \underbrace{[\mathbf{B} \quad \mathbf{AB} \quad \ldots \quad \mathbf{A}^{m-1}\mathbf{B}]}_{\mathbf{Q}} \tag{3.8}$$

where m is the McMillan degree of $\mathbf{G}_m(z)$. By using $\widehat{\mathbf{C}}\widehat{\mathbf{A}}^L = \mathbf{0}$ and $\mathbf{A}^K\mathbf{B} = \mathbf{0}$ it follows that the only nonzero element of this matrix product is the $M \times M$ block matrix $\widehat{\mathbf{C}}\widehat{\mathbf{A}}^{L-1}\mathbf{A}^{K-1}\mathbf{B}$ [i.e., $\mathbf{h}(L)\mathbf{g}(K)$, by (3.7)] which will appear on the top left corner. However by minimality we know that $\mathbf{P}$ and $\mathbf{Q}$ have full column-rank and row-rank respectively ($= m$) so that the above product will be zero if and only if $\widehat{\mathbf{A}}^{L-1}\mathbf{A}^{K-1} = \mathbf{0}$. Thus $\mathbf{h}(L)\mathbf{g}(K) = \mathbf{0}$ if and only if $\widehat{\mathbf{A}}^{L-1}\mathbf{A}^{K-1} = \mathbf{0}$. $\nabla\nabla\nabla$

4. FACTORIZATION OF THE BIORTHONORMAL LAPPED TRANSFORM (BOLT)

The lapped orthogonal transform (*LOT*) was introduced in [7] and further studied in [6] and [8]. The *LOT* is essentially an M channel maximally decimated analysis bank, in which the polyphase matrix satisfies two properties: first, it is a first-order causal FIR system, that is,

$$\mathbf{G}(z) = \mathbf{g}(0) + z^{-1}\mathbf{g}(1) \tag{4.1}$$

(i.e., $\mathbf{E}(z)$ in Fig. 1.1(b) has the above form). Second, it is paraunitary, that is,

$$\mathbf{G}^{-1}(z) = \widetilde{\mathbf{G}}(z) = \mathbf{g}^{\dagger}(0) + z\mathbf{g}^{\dagger}(1).$$

The inverse, therefore, is anticausal FIR. Though $\mathbf{G}(z)$ is a first order system (i.e., the highest power of z^{-1} is z^{-1}), its degree is equal to the rank of $\mathbf{g}(1)$.

A generalization of this to the biorthonormal case would result if we restrict $\mathbf{G}(z)$ above to be merely FIR with an anticausal FIR inverse, and remove the paraunitary (orthonormal) constraint. The inverse is not necessarily equal to $\widetilde{\mathbf{G}}(z)$ anymore. We will call this system the biorthonormal lapped transform (*BOLT*).

By definition the *BOLT* is a maximally decimated analysis bank (Fig. 1), where the polyphase matrix $\mathbf{E}(z)$ is a first order causal FIR transfer matrix (i.e., as in (4.1)), and has anticausal FIR inverse $\mathbf{G}^{-1}(z)$. We sometimes say that $\mathbf{G}(z)$ is a *BOLT* matrix. Clearly the *LOT* is a special case of the *BOLT*. Unlike the *LOT*, the anticausal FIR inverse of the *BOLT* could have higher order. Here is an example:

$$\mathbf{G}(z) = \begin{bmatrix} z^{-1} & -1+z^{-1} & 0 \\ 0 & 1 & 0 \\ -1+z^{-1} & 0 & z^{-1} \end{bmatrix}, \quad \mathbf{G}^{-1}(z) = \begin{bmatrix} z & -1+z & 0 \\ 0 & 1 & 0 \\ -z+z^2 & 1-2z+z^2 & z \end{bmatrix} \tag{4.2}$$

where $\mathbf{G}(z)$ has order $= 1$ and the FIR anticausal inverse has order $= 2$. However the degree of $\mathbf{G}^{-1}(z)$ in z is still equal to the degree of $\mathbf{G}(z)$ in z^{-1} (see Observation 5 at the end of Sec. 5.1 in [1]).

Let $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ and $(\widehat{\mathbf{A}}, \widehat{\mathbf{B}}, \widehat{\mathbf{C}}, \widehat{\mathbf{D}})$ be minimal realizations of $\mathbf{G}(z)$ and its anticausal FIR inverse $\mathbf{G}^{-1}(z)$, related as usual (i.e., Eq. (3.1)). Thus $\mathbf{g}(0) = \mathbf{D}$, and $\mathbf{g}(1) = \mathbf{CB}$. From the structure shown in Fig. 4.1 we see that $\mathbf{A} = \mathbf{0}$ for any minimal realization of $\mathbf{G}(z)$. So any vector $\mathbf{s}$ satisfies $\mathbf{As} = \mathbf{0}$. Next, all the eigenvalues of $\widehat{\mathbf{A}}$ are zero, and there exists $\mathbf{t}^\dagger \neq \mathbf{0}$ satisfying $\mathbf{t}^\dagger \widehat{\mathbf{A}} = \mathbf{0}$. Thus, we can always find vectors $\mathbf{t}$ and $\mathbf{s}$ satisfying (3.6). By using Theorem 3.2 we conclude that the degree reduction step will succeed.† The reduced remainder function will continue to satisfy $\mathbf{A} = \mathbf{0}$ so that we can repeat the degree reduction. We therefore have:

Theorem 4.1. *BOLT factorization.* Consider an M-channel maximally decimated filter bank with analysis bank polyphase matrix $\mathbf{G}(z) = \mathbf{g}(0) + z^{-1}\mathbf{g}(1)$. Suppose this has an FIR anticausal inverse. Then we can factorize $\mathbf{G}(z)$ as

$$\mathbf{G}(z) = \mathbf{V}_\rho(z)\mathbf{V}_{\rho-1}(z)\ldots\mathbf{V}_1(z)\mathbf{G}_0, \tag{4.3}$$

that is, as in Fig. 4.2 where

1. ρ is the McMillan degree of $\mathbf{G}(z)$ (i.e., $\rho =$ the rank of the $M \times M$ matrix $\mathbf{g}(1)$),
2. $\mathbf{V}_m(z) = \mathbf{I} - \mathbf{u}_m\mathbf{v}_m^\dagger + z^{-1}\mathbf{u}_m\mathbf{v}_m^\dagger$ with $\mathbf{v}_m^\dagger\mathbf{u}_m = 1$, and
3. $\mathbf{G}_0 = \mathbf{G}(1)$ and is nonsingular. ◇

Comments.

1. Conversely, a product of the form (4.3) represents a causal FIR system with anticausal FIR inverse, but it may not be *BOLT*. This is because in general the product does not have the form (4.1) but can have higher terms, e.g., $z^{-2}\mathbf{g}(2)$. In the next section we will show how to further constrain the parameters of (4.3) which will ensure that the product is *BOLT*.
2. If $\mathbf{G}(z)$ has real coefficients it can be verified that the coefficients of $\mathbf{V}_m(z)$ are also real.

The most general degree-one *cafacafi* system. A degree one system also has order $= 1$. So a degree one *cafacafi* is a *BOLT* and can be factorized as above, with $\rho = 1$. So we can express it in the form

$$\mathbf{G}(z) = \left(\mathbf{I} - \mathbf{u}\mathbf{v}^\dagger + z^{-1}\mathbf{u}\mathbf{v}^\dagger\right)\mathbf{G}_0, \tag{4.4}$$

where $\mathbf{u}^\dagger\mathbf{v} = 1$ and $\mathbf{G}_0$ is nonsingular (in fact $\mathbf{G}_0 = \mathbf{G}(1)$). So the above equation represents the most general degree-one *cafacafi*. The matrix $\mathbf{G}_0$ has M^2 elements and each of the vectors $\mathbf{u}$ and $\mathbf{v}$ has M

† Note that since $\mathbf{A} = \mathbf{0}$, the quantity $\widehat{\mathbf{A}}^{L-1}\mathbf{A}^{K-1} = \mathbf{0}$ in Theorem 3.3, and yet the factorization succeeds. This is because $\widehat{\mathbf{A}}^{L-1}\mathbf{A}^{K-1} \neq \mathbf{0}$ is only a sufficient but not necessary condition.

elements. Since these elements are constrained by the equation $\mathbf{u}^\dagger \mathbf{v} = 1$, the number of degrees of freedom in the above equation is equal to $(2M-1) + M^2$.

Smith-McMillan form. With ρ denoting the degree, it can be shown that the Smith-McMillan form of $\mathbf{G}(z)$ is $\mathbf{\Lambda}(z) = \begin{bmatrix} z^{-1}\mathbf{I}_\rho & \mathbf{0} \\ \mathbf{0} & \mathbf{I}_{M-\rho} \end{bmatrix}$. This follows from the fact that the quantities ℓ_i defined in Theorem 5.2 of [1] satisfy $0 \leq \ell_i \leq 1$.

A second cascaded realization

The $M \times M$ building block $\mathbf{V}_m(z)$ and the factorization (4.3) can be rewritten in a form which makes the *cafacafi* ɪ ·perty obvious by inspection. To obtain this, let $\mathbf{u}_i$, $0 \leq i \leq M-2$ be a set of mutually orthogonal vectors, which in turn are orthogonal to $\mathbf{v}$. We see that $\mathbf{V}(z)\mathbf{u}_i = \mathbf{u}_i$. (The subscript on $\mathbf{V}(z)$ is dropped for simplicity.) We also see that $\mathbf{V}(z)\mathbf{u} = z^{-1}\mathbf{u}$. Thus

$$\mathbf{V}(z)\left[\mathbf{u}_0 \quad \mathbf{u}_1 \quad \ldots \quad \mathbf{u}_{M-2} \quad \mathbf{u}\right] = \underbrace{\left[\mathbf{u}_0 \quad \mathbf{u}_1 \quad \ldots \quad \mathbf{u}_{M-2} \quad \mathbf{u}\right]}_{\text{call this } \mathbf{T}} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & z^{-1} \end{bmatrix} \tag{4.5}$$

Since $\mathbf{v}^\dagger \mathbf{u} = 1$, the vector $\mathbf{u}$ is not in the span of the $\mathbf{u}_i$'s. So the matrix $\mathbf{T}$ is nonsingular and we can rewrite the above as

$$\mathbf{V}(z) = \mathbf{T} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & z^{-1} \end{bmatrix} \mathbf{T}^{-1} \tag{4.6}$$

Thus, the general form of the degree-one *cafacafi* in Eq. (4.4) can be rewritten as

$$\mathbf{G}(z) = \mathbf{T} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & z^{-1} \end{bmatrix} \mathbf{S} \tag{4.7}$$

where $\mathbf{T}$ and $\mathbf{S}$ are nonsingular matrices. So the BOLT factorization Eq. (4.3) can be rewritten as

$$\mathbf{G}(z) = \mathbf{T}_\rho \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & z^{-1} \end{bmatrix} \mathbf{T}_{\rho-1} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & z^{-1} \end{bmatrix} \ldots \mathbf{T}_1 \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & z^{-1} \end{bmatrix} \mathbf{T}_0 \tag{4.8}$$

where $\mathbf{T}_i$ are nonsingular matrices. The structure is shown in Fig. 4.3. The previous factorization (4.3) has only $2M\rho + M^2$ parameters which is less than the number of matrix elements $M^2\rho + M^2$ in (4.8). So there is some redundancy in the representation (4.8), but its advantage is that it is explicitly clear that the inverse is anticausal FIR.

5. MORE RESULTS ON FIRST ORDER SYSTEMS AND BIORTHONORMAL LAPPED TRANSFORMS

Consider a first order $M \times M$ transfer matrix of the form

$$\mathbf{G}(z) = \mathbf{g}(0) + z^{-1}\mathbf{g}(1). \tag{5.1}$$

Let the McMillan degree be ρ (i.e., the rank of $\mathbf{g}(1)$ is ρ). Suppose $\mathbf{G}(1)$ is nonsingular (as is the case when there exists an FIR inverse, since the determinant would then be a delay). We can then rewrite

$\mathbf{G}(z) = \mathbf{G}(1)\mathbf{F}(z)$ where $\mathbf{F}(1) = \mathbf{I}$. So we can write

$$\mathbf{F}(z) = \mathbf{I} - \mathcal{U}\mathcal{V}^\dagger + z^{-1} \underbrace{\mathcal{U}}_{M\times\rho} \underbrace{\mathcal{V}^\dagger}_{\rho\times M} \tag{5.2}$$

with the constant matrices $\mathcal{U}$ and $\mathcal{V}^\dagger$ having rank ρ. We will now relate the properties of the inverse $\mathbf{F}^{-1}(z)$ to the properties of the matrices $\mathcal{U}\mathcal{V}^\dagger$ and $\mathcal{V}^\dagger\mathcal{U}$. Such a study adds significantly to the understanding of the biorthonormal lapped transform.

5.1. Inverse of the first order system $\left(\mathbf{I} - \mathcal{U}\mathcal{V}^\dagger + z^{-1}\mathcal{U}\mathcal{V}^\dagger\right)$

The nature of the inverse of (5.2) depends largely on the properties of the $\rho \times \rho$ matrix $\mathcal{V}^\dagger\mathcal{U}$ as shown by the results to be developed below.

Lemma 5.1. Consider the system $\mathbf{F}(z) = \mathbf{I} - \mathcal{U}\mathcal{V}^\dagger + z^{-1}\mathcal{U}\mathcal{V}^\dagger$ where $\mathcal{U}$ and $\mathcal{V}$ are $M \times \rho$ with rank ρ (so that $\mathbf{F}(z)$ has degree ρ). There exists an anticausal inverse for this system if and only if $\mathcal{V}^\dagger\mathcal{U}$ (which is $\rho \times \rho$) is nonsingular. ◊

Proof. Fig. 5.1 shows an implementation of $\mathbf{F}(z)$ with ρ delays, i.e., a minimal implementation. The state space description $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ for this is

$$\mathbf{A} = \mathbf{0},\ \ \mathbf{B} = \mathcal{V}^\dagger,\ \ \mathbf{C} = \mathcal{U},\ \ \mathbf{D} = \mathbf{I} - \mathcal{U}\mathcal{V}^\dagger \tag{5.3}$$

The realization matrix $\mathcal{R}$ is then

$$\mathcal{R} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{array}{c} \\ \rho \\ M \end{array} \!\!\begin{array}{c} \begin{array}{cc} \rho & \quad M \end{array} \\ \begin{pmatrix} \mathbf{0} & \mathcal{V}^\dagger \\ \mathcal{U} & \mathbf{I} - \mathcal{U}\mathcal{V}^\dagger \end{pmatrix} \end{array} \tag{5.4}$$

Recall from Theorem 5.1 of [1] that there exists an anticausal inverse if and only if the above matrix is nonsingular. We will show that this matrix is nonsingular if and only if $\mathcal{V}^\dagger\mathcal{U}$ is nonsingular. Suppose $\mathcal{R}\mathbf{x} = \mathbf{0}$ for some vector $\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix}$. Then

$$\mathcal{V}^\dagger\mathbf{x}_2 = \mathbf{0}, \quad \text{and} \quad \mathcal{U}\mathbf{x}_1 + \mathbf{x}_2 = \mathbf{0}. \tag{5.5}$$

Combining these two equations we get $\mathcal{V}^\dagger\mathcal{U}\mathbf{x}_1 = \mathbf{0}$. If $\mathcal{V}^\dagger\mathcal{U}$ is nonsingular then $\mathbf{x}_1 = \mathbf{0}$ and so $\mathbf{x}_2 = -\mathcal{U}\mathbf{x}_1 = \mathbf{0}$ from (5.5). This implies that if $\mathcal{R}\mathbf{x} = \mathbf{0}$ then $\mathbf{x}$ is necessarily $\mathbf{0}$. So $\mathcal{R}$ is nonsingular.

On the other hand, if $\mathcal{V}^\dagger\mathcal{U}$ is singular there exists $\mathbf{y} \neq \mathbf{0}$ such that $\mathcal{V}^\dagger\mathcal{U}\mathbf{y} = \mathbf{0}$. If we now choose $\mathbf{x}_1 = -\mathbf{y}$ and $\mathbf{x}_2 = \mathcal{U}\mathbf{y}$, then $\mathcal{R}$ is annihilated by $\mathbf{x}$ proving that it is singular. So $\mathcal{R}$ is nonsingular if and only if $\mathcal{V}^\dagger\mathcal{U}$ is nonsingular. This completes the proof. ▽▽▽

As an example suppose

$$\mathcal{U}\mathcal{V}^\dagger = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \quad -1] \tag{5.6}$$

Then $\rho = 1$ but $\mathcal{V}^\dagger\mathcal{U} = 0$, so there does not exist an anticausal inverse. As another example suppose $\mathcal{U}\mathcal{V}^\dagger$ itself is nonsingular (i.e., $\rho = M$); then $\mathcal{V}^\dagger\mathcal{U}$ is nonsingular and there exists an anticausal inverse, possibly IIR. The next theorem makes precise the conditions under which the inverses are FIR.

Theorem 5.1. Consider the first order system $\mathbf{F}(z) = \mathbf{I} - \mathcal{U}\mathcal{V}^\dagger + z^{-1}\mathcal{U}\mathcal{V}^\dagger$ where $\mathcal{U}$ and $\mathcal{V}$ are $M \times \rho$ with rank ρ (so $\mathcal{U}\mathcal{V}^\dagger$ has rank ρ and $\mathbf{F}(z)$ has degree ρ). Then the inverse of $\mathbf{F}(z)$ is

1. FIR if and only if all eigenvalues of $\mathcal{U}\mathcal{V}^\dagger$ are restricted to be 0's and 1's.
2. FIR and anticausal (i.e., $\mathbf{F}(z)$ is *cafacafi*) if and only if $\mathcal{U}\mathcal{V}^\dagger$ has ρ of its eigenvalues equal to unity and the remaining $M - \rho$ eigenvalues equal to zero.
3. FIR and causal (i.e., $\mathbf{F}(z)$ is unimodular in z^{-1}) if and only if $\mathcal{U}\mathcal{V}^\dagger$ has all eigenvalues equal to zero. $\diamond$

Comments.

1. Restricting the eigenvalues of a matrix $\mathbf{P}$ to be zeros and ones does not imply that $\mathbf{P}^2 = \mathbf{P}$ or that it is a projection matrix.† For example, the matrix $\mathcal{U}\mathcal{V}^\dagger$ in (5.6) has all eigenvalues $= 0$, but $\mathbf{P}^2 = \mathbf{0} \neq \mathbf{P}$.
2. Since $\mathcal{U}\mathcal{V}^\dagger$ has rank ρ it can have at most ρ nonzero eigenvalues. But it could be fewer, as in the extreme example of a triangular matrix with all diagonal elements equal to zero. Another example is (5.6) which has rank $= 1$, but all the eigenvalues are equal to zero.

Proof of Theorem 5.1. From the unitary triangularization theorem [11] we can write $\mathcal{U}\mathcal{V}^\dagger = \mathbf{T}\boldsymbol{\Delta}\mathbf{T}^\dagger$ where $\mathbf{T}\mathbf{T}^\dagger = \mathbf{I}$, and $\boldsymbol{\Delta}$ is upper triangular with the eigenvaules $\{\lambda_0, \lambda_1 \ldots \lambda_{\rho-1}, 0, \ldots 0\}$ on the diagonals. (Since the rank is ρ there could be at most ρ nonzero eigenvalues). We can then express

$$\mathbf{F}(z) = \mathbf{T}\Big(\mathbf{I} - \boldsymbol{\Delta} + z^{-1}\boldsymbol{\Delta}\Big)\mathbf{T}^\dagger \tag{5.7}$$

so that

$$\det \mathbf{F}(z) = \prod_{i=0}^{\rho-1}\Big(1 - \lambda_i + z^{-1}\lambda_i\Big). \tag{5.8}$$

This is of the form cz^{-K} (which is necessary and sufficient for the existence of an FIR inverse) if and only if $\lambda_i = 0$ or 1 for each i. Since the degree of $\mathbf{F}(z)$ is ρ, the FIR inverse is anticausal if and only if the determinant is $cz^{-\rho}$ (Theorem 5.3, [1]). This will be the case if and only if $\mathcal{U}\mathcal{V}^\dagger$ has ρ eigenvalues equal to unity (and, of course, the remaining $M - \rho$ eigenvalues $= 0$). Finally the FIR inverse is causal (i.e., $\mathbf{F}(z)$ is unimodular) if and only if the determinant is a constant, that is $\lambda_i = 0$ for all i. $\nabla\nabla\nabla$

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† A matrix $\mathbf{P}$ is said to be a projection if it is Hermitian and $\mathbf{P}^2 = \mathbf{P}$ (p. 75, [10]).

We can combine Lemma 5.1 and Theorem 5.1 and restate everything in terms of $\mathcal{V}^\dagger\mathcal{U}$ rather than $\mathcal{U}\mathcal{V}^\dagger$ as follows.

Theorem 5.2. Consider the system $\mathbf{F}(z) = \mathbf{I} - \mathcal{U}\mathcal{V}^\dagger + z^{-1}\mathcal{U}\mathcal{V}^\dagger$ where $\mathcal{U}$ and $\mathcal{V}$ are $M \times \rho$ with rank ρ (so $\mathcal{U}\mathcal{V}^\dagger$ has rank ρ and $\mathbf{F}(z)$ has degree ρ). Then

1. $\mathbf{F}(z)$ has an anticausal inverse if and only if $\mathcal{V}^\dagger\mathcal{U}$ is nonsingular.
2. The inverse of $\mathbf{F}(z)$ is FIR if and only if all eigenvalues of $\mathcal{V}^\dagger\mathcal{U}$ are restricted to be 0's and 1's.
3. The inverse of $\mathbf{F}(z)$ is FIR and anticausal (i.e., $\mathbf{F}(z)$ is *cafacafi*) if and only if $\mathcal{V}^\dagger\mathcal{U}$ has all eigenvalues equal to unity.
4. The inverse of $\mathbf{F}(z)$ is FIR and causal (i.e., $\mathbf{F}(z)$ is unimodular) if and only if $\mathcal{V}^\dagger\mathcal{U}$ has all eigenvalues equal to zero. $\diamond$

Proof. Part 1 is a repetition of Lemma 5.1. Parts 2 and 4 follow from Theorem 5.1 by using the fact that every nonzero eigenvalue of the matrix $\mathbf{PQ}$ is an eigenvalue of $\mathbf{QP}$ (for any two matrices $\mathbf{P}$ and $\mathbf{Q}$ for which $\mathbf{PQ}$ and $\mathbf{QP}$ are defined). Part 3 follows by combining parts 1 and 2; indeed, the nonsingularity of $\mathcal{V}^\dagger\mathcal{U}$ and the condition that the eigenvalues be restricted to be ones and zeros is equivalent to the statement that all the eigenvalues of $\mathcal{V}^\dagger\mathcal{U}$ are equal to unity. $\nabla\nabla\nabla$

Example 5.1. The cases where $\mathcal{V}^\dagger\mathcal{U} = \mathbf{I}_\rho$ and $\mathcal{V}^\dagger\mathcal{U} = \mathbf{0}$ give examples of FIR systems with anticausal and causal FIR inverses respectiveley. We have

$$\left(\mathbf{I} - \mathcal{U}\mathcal{V}^\dagger + z^{-1}\mathcal{U}\mathcal{V}^\dagger\right)^{-1} = \begin{cases} \mathbf{I} - \mathcal{U}\mathcal{V}^\dagger + z\mathcal{U}\mathcal{V}^\dagger & \text{for } \mathcal{V}^\dagger\mathcal{U} = \mathbf{I}_\rho \\ \mathbf{I} + \mathcal{U}\mathcal{V}^\dagger - z^{-1}\mathcal{U}\mathcal{V}^\dagger & \text{for } \mathcal{V}^\dagger\mathcal{U} = \mathbf{0} \end{cases} \tag{5.9}$$

as one can verify by direct multiplication. Notice that if $\mathcal{V}^\dagger\mathcal{U} = \mathbf{I}_\rho$ then the inverse is also of first order. So first order *cafacafi* systems with higher order inverses (as in (4.2)) are not covered by the system with $\mathcal{V}^\dagger\mathcal{U} = \mathbf{I}_\rho$. In Sec. 2.1 we saw the special case where $\rho = 1$ (i.e., $\mathcal{U}$ and $\mathcal{V}$ were vectors with $\mathcal{V}^\dagger\mathcal{U} = 1$ and 0 respectively).

Example 5.2. The following example

$$\mathcal{U}\mathcal{V}^\dagger = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \mathcal{V}^\dagger\mathcal{U} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

satisfies part 4 of Theorem 5.2 so that $\mathbf{F}(z)$ is unimodular, even though $\mathcal{V}^\dagger\mathcal{U} \neq \mathbf{0}$.

Example 5.3. As a special case consider $\mathbf{I} - \mathbf{P} + z^{-1}\mathbf{P}$ where $\mathbf{P}^2 = \mathbf{P}$. With ρ denoting the rank of $\mathbf{P}$, we can write $\mathbf{P} = \mathcal{U}\mathcal{V}^\dagger$. Now $\mathbf{P}^2 = \mathbf{P}$ implies $\mathcal{U}\mathcal{V}^\dagger\mathcal{U}\mathcal{V}^\dagger = \mathcal{U}\mathcal{V}^\dagger$. Premultiplying by $\mathcal{U}^\dagger$ and postmultiplying with $\mathcal{V}$ and using the facts that $\mathcal{U}^\dagger\mathcal{U}$ and $\mathcal{V}^\dagger\mathcal{V}$ are nonsingular we obtain $\mathcal{V}^\dagger\mathcal{U} = \mathbf{I}$. From part 3 of Theorem

5.2 we therefore conclude that there exists an anticausal FIR inverse for $\mathbf{I}-\mathbf{P}+z^{-1}\mathbf{P}$, when $\mathbf{P}^2=\mathbf{P}$. In fact the inverse is $\mathbf{I}-\mathbf{P}+z\mathbf{P}$, as can be verified by direct substitution.

Example 5.4. Unimodular system. By a slight modification of the above theorem we can show that $\mathbf{I}+z^{-1}\mathcal{U}\mathcal{V}^\dagger$ is unimodular if and only if $\mathcal{V}^\dagger\mathcal{U}$ has all eigenvalues equal to zero.

5.2. General Expression and Complete Parameterization of the Biorthonormal Lapped Transforms (BOLT)

In Sec. 4 we considered the biorthonormal lapped transforms or *BOLT* systems. These are first order *cafacafi* systems, that is, systems of the form $\mathbf{G}(z)=\mathbf{g}(0)+z^{-1}\mathbf{g}(1)$ with anticausal FIR inverses. Since this implies $\mathbf{G}(1)$ is nonsingular, we can write $\mathbf{G}(z)=\mathbf{G}(1)\mathbf{F}(z)$ where $\mathbf{F}(z)$ is as in (5.2). Using Theorem 5.2 (part 3) we can say that a system $\mathbf{G}(z)$ is *BOLT* if and only if it has the form $\mathbf{G}(z)=\mathbf{G}(1)(\mathbf{I}-\mathcal{U}\mathcal{V}^\dagger+z^{-1}\mathcal{U}\mathcal{V}^\dagger)$ where $\mathcal{V}^\dagger\mathcal{U}$ has all eigenvalues equal to unity.

In Sec. 4 we showed that the *BOLT* can be factorized as in (4.3) where

$$\mathbf{V}_m(z)=\mathbf{I}-\mathbf{u}_m\mathbf{v}_m^\dagger+z^{-1}\mathbf{u}_m\mathbf{v}_m^\dagger,\quad \mathbf{v}_m^\dagger\mathbf{u}_m=1, \tag{5.10}$$

and ρ is the degree of $\mathbf{G}(z)$ (i.e., $\rho=$ rank of $\mathbf{g}(1)$). Conversely, if we have a product of the form (4.3) with $\mathbf{V}_m(z)$ as above, it still represents a system with anticausal FIR inverse, but may have order >1 (i.e., there could be terms $\mathbf{g}(n)z^{-n}, n>1$ in $\mathbf{G}(z)$). To ensure that the product has order $=1$ (i.e., that it represents a *BOLT*), we need to impose further restrictions on $\mathbf{u}_k$ and $\mathbf{v}_k$. Suppose we restrict these vectors to be such that

$$\mathbf{v}_k^\dagger\mathbf{u}_i=\begin{cases}0, & 1\le i\le k-1,\\ 1, & i=k\end{cases} \tag{5.11}$$

Then it is easily verifed by induction that the product

$$\mathbf{G}(z)=\mathbf{G}(1)\mathbf{V}_\rho(z)\mathbf{V}_{\rho-1}(z)\ldots\mathbf{V}_1(z), \tag{5.12}$$

with $\mathbf{V}_m(z)$ defined as above does reduce to the form

$$\mathbf{G}(z)=\mathbf{G}(1)\Big(\mathbf{I}-\mathcal{U}\mathcal{V}^\dagger+z^{-1}\mathcal{U}\mathcal{V}^\dagger\Big), \tag{5.13}$$

with the constant matrices $\mathcal{V}$ and $\mathcal{U}$ given by

$$\mathcal{V}=[\mathbf{v}_1\quad \mathbf{v}_2\quad \ldots\quad \mathbf{v}_\rho],\quad \mathcal{U}=[\mathbf{u}_1\quad \mathbf{u}_2\quad \ldots\quad \mathbf{u}_\rho]. \tag{5.14}$$

Notice that the constant matrix $\mathbf{G}(1)$ occurs as the left-most factor unlike in (4.3). This difference is immaterial; a slight variation of the steps would lead to the form (4.3). Except for this difference, the structure for (5.12) is as in Fig. 4.2.

Conversely, can we represent any *BOLT* system as in (5.12) with the restriction (5.11)? The answer is in the affirmative: If $\mathbf{G}(z)$ is *BOLT*, this means in particular that it has an FIR inverse, and so $\mathbf{G}(1)$ is nonsingular. So we can always write a degree ρ BOLT as in (5.13), where $\mathcal{U}$ and $\mathcal{V}$ are $M \times \rho$ matrices with rank ρ. Now $\mathcal{U}\mathcal{V}^\dagger = \mathcal{U}\mathbf{T}\mathbf{T}^\dagger\mathcal{V}^\dagger$ for any unitary $\mathbf{T}$, and we can rewrite $\mathcal{U}\mathcal{V}^\dagger = \mathcal{U}_1\mathcal{V}_1^\dagger$ by defining $\mathcal{U}_1 = \mathcal{U}\mathbf{T}$ and $\mathcal{V}_1^\dagger = \mathbf{T}^\dagger\mathcal{V}^\dagger$. Note that $\mathcal{V}_1^\dagger\mathcal{U}_1 = \mathbf{T}^\dagger\mathcal{V}^\dagger\mathcal{U}\mathbf{T}$. By proper choice of $\mathbf{T}$ we can ensure that $\mathcal{V}_1^\dagger\mathcal{U}_1$ is a triangular matrix. In other words, we can assume without loss of generality that $\mathcal{V}^\dagger\mathcal{U}$ is triangular. Since $\mathbf{G}(z)$ is *cafacafi* we see that this matrix has all diagonal elements equal to unity (use part 3 of Theorem 5.2), that is,

$$\mathcal{V}^\dagger\mathcal{U} = \begin{bmatrix} 1 & \times & \times & \dots & \times \\ 0 & 1 & \times & \dots & \times \\ 0 & 0 & 1 & \dots & \times \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 1 \end{bmatrix} \tag{5.15}$$

where $\times$ stands for possibly nonzero elements. Now denote the columns of $\mathcal{V}$ and $\mathcal{U}$ as in (5.14). Then the property (5.15) means that (5.11) is satisfied.

Thus we have defined a set of vectors $\mathbf{u}_m$ and $\mathbf{v}_m$, $1 \le m \le \rho$ such that they satisfy (5.11). We already mentioned that if such $\mathbf{u}_m$ and $\mathbf{v}_m$ are used in the product (5.12), the result has the form (5.13) with $\mathcal{U}$ and $\mathcal{V}$ given by (5.14). In other words, the given BOLT matrix (5.13) can indeed be represented as in (5.12), with the vectors satisfying (5.11).

We can summarize all of the above results as follows.

Theorem 5.3. *BOLT Characterization.* Consider an $M \times M$ transfer matrix $\mathbf{G}(z)$. We say that this is a *BOLT* if $\mathbf{G}(z) = \mathbf{g}(0) + z^{-1}\mathbf{g}(1)$, and it has and anticausal FIR inverse. The following statements are equivalent:

1. $\mathbf{G}(z)$ is a *BOLT*.
2. $\mathbf{G}(z)$ can be factorized as $\mathbf{G}(z) = \mathbf{G}(1)\mathbf{V}_\rho(z)\mathbf{V}_{\rho-1}(z)\dots\mathbf{V}_1(z)$ where $\mathbf{G}(1)$ is nonsingular and $\mathbf{V}_m(z)$ are as in (5.10), with the vectors $\mathbf{v}_k$ and $\mathbf{u}_i$ satisfying (5.11).
3. $\mathbf{G}(z)$ can be written in the form $\mathbf{G}(z) = \mathbf{G}(1)(\mathbf{I} - \mathcal{U}\mathcal{V}^\dagger + z^{-1}\mathcal{U}\mathcal{V}^\dagger)$, where $\mathbf{G}(1)$ is nonsingular and $\mathcal{V}^\dagger\mathcal{U}$ has all eigenvalues equal to unity.
4. $\mathbf{G}(z)$ can be written in the form $\mathbf{G}(z) = \mathbf{G}(1)(\mathbf{I} - \mathcal{U}\mathcal{V}^\dagger + z^{-1}\mathcal{U}\mathcal{V}^\dagger)$, where $\mathbf{G}(1)$ is nonsingular and $\mathcal{V}^\dagger\mathcal{U}$ has the form (5.15).

Thus if $\mathbf{G}(z)$ is *BOLT* with $\mathcal{V}^\dagger\mathcal{U}$ written in the form (5.15), the columns of $\mathcal{V}$ and $\mathcal{U}$ [see (5.14)] satisfy (5.11), and can be taken to be the vectors $\mathbf{v}_m$ and $\mathbf{u}_m$ in the factorization (5.12). So the factorization is determined simply by identifying the columns of $\mathcal{V}$ and $\mathcal{U}$. ◇

Degrees of freedom. Thus the *BOLT* is characterized by (5.12) which has a nonsingular matrix $\mathbf{G}(1)$ with M^2 elements, and 2ρ vectors $\mathbf{u}_k, \mathbf{v}_k$, with M elements each. That is, there are $M^2 + 2\rho M$ scalar elements associated with the expression (5.12). But the number of freedoms is less than this, in view of the constraints (5.11). In the real coefficient case it can be verified that the $M \times M$ degree-ρ *BOLT* has $M^2 + 2\rho M - 0.5\rho(\rho+1)$ degrees of freedom. In the special case of the *LOT* (i.e., the paraunitary case) we have $\mathbf{u}_k = \mathbf{v}_k$ and $\mathbf{G}(1)$ is unitary so there are only $0.5M(M-1) + \rho M - 0.5\rho(\rho+1)$ degrees of freedom. Traditional transform coding (which is a special case where $\mathcal{U}\mathcal{V}^\dagger = \mathbf{0}$ and $\mathbf{G}(1)$ is unitary) has only $0.5M(M-1)$ freedoms. The extra freedom offered by the *BOLT* can perhaps be exploited to obtain better attenuation for the analysis filters (see example below), or to impose other constraints such as linear phase, regularity (for wavelet synthesis [12]) and so forth. This topic requires detailed investigation, and is beyond the scope of this paper.

Design Example 5.1: The BOLT filter bank

We now present a design example for the *BOLT* filter bank.† Let $M = 8$ [i.e., an eight channel filter bank, see Fig. 1.1(a)]. Let the polyphase matrix $\mathbf{E}(z) = \mathbf{g}(0) + z^{-1}\mathbf{g}(1)$ with rank of $\mathbf{g}(1)$ equal to three (i.e., degree of $\mathbf{E}(z)$ is three). This is constrained to be a *BOLT* by expressing it in the factored form (5.12) and constraining the vectors to satisfy (5.11). Under these constraints, the magnitude responses $|H_k(e^{j\omega})|$ of the analysis filters are optimized. The result is shown in Fig. 5.2 (a). For comparison, Fig. 5.2(b) shows the responses of the corresponding *LOT* filter bank (i.e., with the vectors further constrained such that $\mathbf{v}_i = \mathbf{u}_i$ for each i). The improved filtering characteristics of the *BOLT* over the *LOT* is clear from the plots.

6. DEGREE-TWO DYADIC BUILDING BLOCKS

If the degree-one reduction scheme of Sec. 2 has to work, there should exist vectors $\mathbf{u}$ and $\mathbf{v}$ such that (2.10) holds. If this is not the case, one might consider extracting the building block from the right rather than left, i.e., one might try the decomposition $\mathbf{G}_m(z) = \mathbf{G}_{m-1}(z)\mathbf{V}_m(z)$ instead of (2.6). In this case the degree reduction equations remain the same except that $\mathbf{g}_m(0)$ and $\mathbf{h}_m(0)$ are interchanged. Thus, degree-one reduction will fail when neither of the following two conditions

$$\text{Condition 1:} \quad \mathbf{v}^\dagger\mathbf{g}_m(0) = \mathbf{0}, \quad \mathbf{h}_m(0)\mathbf{u} = \mathbf{0}, \quad \mathbf{u}^\dagger\mathbf{v} = 1, \tag{6.1}$$

$$\text{Condition 2:} \quad \mathbf{v}^\dagger\mathbf{h}_m(0) = \mathbf{0}, \quad \mathbf{g}_m(0)\mathbf{u} = \mathbf{0}, \quad \mathbf{u}^\dagger\mathbf{v} = 1 \tag{6.2}$$

can be satisfied for any choice of $\mathbf{u}$ and $\mathbf{v}$.

† We would like to thank Yuan-Pei Lin, graduate student, Caltech, for generating this example.

6.1. Cases where degree-one reduction fails

We can create examples of *cafacafi* systems for which degree-one reduction will fail. For example, let us consider the 2×2 case ($M = 2$). In this case we can exactly specify the conditions when the degree reduction will fail.

Lemma 6.1. Let $\mathbf{G}_m(z)$ be 2×2 *cafacafi* with inverse $\mathbf{H}_m(z)$ (both as in (2.5)). Assume $\mathbf{g}_m(0) \neq \mathbf{0}$ and $\mathbf{h}_m(0) \neq \mathbf{0}$ and $K, L > 0$ in (2.5). Then the degree reduction by one (using the building block (2.1)) will fail if and only if $\mathbf{g}_m(0)\mathbf{h}_m(0) = \mathbf{h}_m(0)\mathbf{g}_m(0) = \mathbf{0}$, that is, if and only if $\mathbf{D}\widehat{\mathbf{D}} = \widehat{\mathbf{D}}\mathbf{D} = \mathbf{0}$ in terms of state space notations (Sec. 3). $\diamond$

Proof. Since the 2×2 matrices $\mathbf{g}_m(0)$ and $\mathbf{h}_m(0)$ are singular (Sec. 5.3 of [1]) and nonzero, the vectors $\mathbf{v}$ and $\mathbf{u}$ satisfying $\mathbf{v}^\dagger \mathbf{h}_m(0) = \mathbf{0}$ and $\mathbf{g}_m(0)\mathbf{u} = \mathbf{0}$ are unique up to scale. Clearly $\mathbf{g}_m(0)$ and $\mathbf{h}_m(0)$ have rank one, and we can write

$$\mathbf{g}_m(0) = \mathbf{a}\mathbf{b}^\dagger \quad \text{and} \quad \mathbf{h}_m(0) = \mathbf{c}\mathbf{d}^\dagger$$

for some 2×1 non null vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ and $\mathbf{d}$. Thus $\mathbf{v}^\dagger \mathbf{h}_m(0) = \mathbf{0}$ and $\mathbf{g}_m(0)\mathbf{u} = \mathbf{0}$ imply, respectively,

$$\mathbf{v}^\dagger \mathbf{c} = 0 \quad \text{and} \quad \mathbf{b}^\dagger \mathbf{u} = 0.$$

From this we see that if $\mathbf{v}^\dagger \mathbf{u} = 0$, then $\mathbf{u} = \mathbf{c}$ and $\mathbf{v} = \mathbf{b}$ (up to scale) and this implies $\mathbf{b}^\dagger \mathbf{c} = \mathbf{v}^\dagger \mathbf{u} = 0$, that is, $\mathbf{g}_m(0)\mathbf{h}_m(0) = \mathbf{0}$. So if (6.2) cannot be satisfied then $\mathbf{g}_m(0)\mathbf{h}_m(0) = \mathbf{0}$. Conversely, let $\mathbf{g}_m(0)\mathbf{h}_m(0) = \mathbf{0}$. Then $\mathbf{b}^\dagger \mathbf{c} = 0$. So the conditions $\mathbf{v}^\dagger \mathbf{h}_m(0) = \mathbf{0}$ and $\mathbf{g}_m(0)\mathbf{u} = \mathbf{0}$ imply, respectively, $\mathbf{v} = \mathbf{b}$ and $\mathbf{u} = \mathbf{c}$ (up to scale) so that $\mathbf{v}^\dagger \mathbf{u} = \mathbf{b}^\dagger \mathbf{c} = 0$, and (6.2) cannot be satisfied. Thus (6.2) cannot be satisfied if and only if $\mathbf{g}_m(0)\mathbf{h}_m(0) = \mathbf{0}$. Similarly (6.1) cannot be satisfied if and only if $\mathbf{h}_m(0)\mathbf{g}_m(0) = \mathbf{0}$. $\nabla\nabla\nabla$

Since the *cafacafi* system and its inverse satisfy the state space relations (3.4) and (3.5), we can restate the above result in terms of state space parameters (Sec. 3) like this: the degree-one reduction step will fail in the 2×2 case if and only if

$$\widehat{\mathbf{C}}\mathbf{B} = \mathbf{C}\widehat{\mathbf{B}} = \mathbf{I}.$$

This follows by setting $\widehat{\mathbf{D}}\mathbf{D} = \mathbf{0}$ in the last equation of (3.5) and $\mathbf{D}\widehat{\mathbf{D}} = \mathbf{0}$ in the last equation of (3.4).

2×2 Example where degree-one reduction fails

Now consider the $M \times M$ system

$$\mathbf{G}_m(z) = \mathbf{a}\mathbf{b}^\dagger + z^{-1}\mathbf{I} + z^{-2}\mathbf{a}\mathbf{b}^\dagger, \quad \mathbf{a}^\dagger \mathbf{b} = 0. \tag{6.3}$$

where $\mathbf{a}$ and $\mathbf{b}$ are non zero $M \times 1$ vectors. We can verify that the inverse is $\mathbf{G}_m^{-1}(z) = -\mathbf{a}\mathbf{b}^\dagger + z\mathbf{I} - \mathbf{a}\mathbf{b}^\dagger z^2$, by multiplying the two expressions. This system is therefore *cafacafi*. Since $\mathbf{a}^\dagger \mathbf{b} = 0$, we have $\mathbf{g}_m(0)\mathbf{h}_m(0) =$

$\mathbf{h}_m(0)\mathbf{g}_m(0) = \mathbf{0}$. So by Lemma 6.1, the degree reduction step will fail in the $M = 2$ case. For $M = 2$, the system (6.3) therefore serves as a *cafacafi* example where the degree-one reduction fails, that is, neither (6.1) nor (6.2) be satisfied for any pair of vectors $\mathbf{u}$ and $\mathbf{v}$.

In Appendix A we show that for arbitrary M the degree of (6.3) is equal to M, and present a minimal implementation (Fig. A.1).

$M \times M$ example where degree-one reduction fails

If $M > 2$, the system in (6.3) is still *cafacafi*, but its degree *can* be reduced successfully by one, using the building block (2.1). To see this note that in this case there exists a vector $\mathbf{w}$ orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ so that we can set $\mathbf{u} = \mathbf{v} = \mathbf{w}$ and satisfy (6.1). To create an $M \times M$ example which *cannot* be factorized into degree-one builing blocks, consider

$$\mathbf{G}(z) = \mathbf{P}\mathbf{P}^\dagger + \mathbf{a}\mathbf{b}^\dagger + z^{-1}(\mathbf{I} - \mathbf{P}\mathbf{P}^\dagger) + z^{-2}\mathbf{a}\mathbf{b}^\dagger \tag{6.4}$$

where $\mathbf{P}$ is $M \times (M-2)$, and $\mathbf{a}$ and $\mathbf{b}$ are column vectors such that $[\mathbf{P} \quad \mathbf{a} \quad \mathbf{b}]$ is unitary. It can then be verified that its inverse is

$$\mathbf{G}^{-1}(z) = \mathbf{P}\mathbf{P}^\dagger - \mathbf{a}\mathbf{b}^\dagger + z(\mathbf{I} - \mathbf{P}\mathbf{P}^\dagger) - z^2\mathbf{a}\mathbf{b}^\dagger, \tag{6.5}$$

which is anticausal FIR. In the notation of (2.5) we have $\mathbf{g}_m(0) = \mathbf{P}\mathbf{P}^\dagger + \mathbf{a}\mathbf{b}^\dagger$ and $\mathbf{h}_m(0) = \mathbf{P}\mathbf{P}^\dagger - \mathbf{a}\mathbf{b}^\dagger$. Both of these matrices have rank $M-1$ (e.g., write $\mathbf{g}_m(0) = [\mathbf{P} \quad \mathbf{a}]\,[\mathbf{P} \quad \mathbf{b}]^\dagger$ and apply Sylvester's inequality [4]) so that the annihilating vectors $\mathbf{u}$ and $\mathbf{v}$ in Eq. (6.1) are unique. In fact the annihilating vectors in (6.1) are $\mathbf{u} = \mathbf{a}$ and $\mathbf{v} = \mathbf{b}$ so that $\mathbf{u}^\dagger\mathbf{v} = 0$. Thus the condition $\mathbf{u}^\dagger\mathbf{v} = 1$ in (6.1) cannot be satisfied. Similarly (6.2) cannot be satisfied. So the degree of (6.4) cannot be reduced by extracting a degree-one *cafacafi* building block.

The degree of $\mathbf{G}(z)$ is clearly ≥ 2 since the order is seen to be two from (6.4). We will show that the degree is exactly two by displaying an implementation with two delays. Since $[\mathbf{P} \quad \mathbf{a} \quad \mathbf{b}]$ is unitary, we have $\mathbf{I}_M = \mathbf{a}\mathbf{a}^\dagger + \mathbf{b}\mathbf{b}^\dagger + \mathbf{P}\mathbf{P}^\dagger$. Using this we can see that the system $\mathbf{G}(z)$ in (6.4) can be implemented as in Fig. 6.1. So the degree of $\mathbf{G}(z)$ is two indeed.

6.2. Degree reduction equations with degree-two building blocks

The fact that we cannot factorize the 2×2 system (6.3) into degree-one blocks leads us to ask if we can factorize a general 2×2 *cafacafi* system using a combination of degee-one and degree-two building blocks. With some algebra it can be shown (Appendix B) that the most general 2×2 *cafacafi* system with degree equal to two, which cannot be factorized into degree one *cafacafi* systems has the form

$$\mathbf{V}_2(z) = \mathbf{u}\mathbf{v}^\dagger + z^{-1}\mathbf{I} + sz^{-2}\mathbf{u}\mathbf{v}^\dagger, \quad \mathbf{u}^\dagger\mathbf{v} = 0, \tag{6.6}$$

where s is a nonzero scalar.† By explicit multiplication we can verify that the inverse is $\mathbf{V}_2^{-1}(z) = -s\mathbf{u}\mathbf{v}^\dagger + z\mathbf{I} - z^2\mathbf{u}\mathbf{v}^\dagger$. Since $\mathbf{V}_2(z)$ is degree-two *cafacafi*, we have $[\det \mathbf{V}_2(z)] = cz^{-2}, \quad c \neq 0$.

Now let $\mathbf{G}_m(z)$ be degree-m *cafacafi* with $[\det \mathbf{G}_m(z)] = c_m z^{-m}$. Suppose degree-one reduction fails (i.e., we cannot find $\mathbf{u}$ and $\mathbf{v}$ satisfying either (6.1) or (6.2)). Suppose we wish to use (6.6) to obtain a degree reduction by two, i.e., we wish to find a degree-$(m-2)$ *cafacafi* system $\mathbf{G}_{m-2}(z)$ such that

$$\mathbf{G}_m(z) = \mathbf{V}_2(z)\mathbf{G}_{m-2}(z). \tag{6.7}$$

Since $[\det \mathbf{V}_2(z)] = cz^{-2}$ we have $[\det \mathbf{G}_{m-2}(z)] = c_{m-2}z^{-(m-2)}$. It can be shown (Appendix C) that $\mathbf{G}_{m-2}(z)$ will be *cafacafi* if and only if $\mathbf{u}$ and $\mathbf{v}$ are such that

$$\mathbf{g}_m(0) = \mathbf{u}\mathbf{v}^\dagger\mathbf{g}_m(1), \quad \mathbf{h}_m(0) = -s\mathbf{h}_m(1)\mathbf{u}\mathbf{v}^\dagger \tag{6.8}$$

where $\mathbf{g}_m(n)$ and $\mathbf{h}_m(n)$ are the impulse response coefficients of $\mathbf{G}_m(z)$ and its inverse respectively (see Eq. (2.5)). If the above can be satisfied by choice of $\mathbf{u}$ and $\mathbf{v}$ then $\mathbf{G}_{m-2}(z)$ is *cafacafi* with degree $m-2$ because its determinant is $c_{m-2}z^{-(m-2)}$ (Theorem 5.3, [1]).

Another example of an irreducible cafacafi system. Consider the 2×2 system $\mathbf{G}_4(z) = \mathbf{a}\mathbf{b}^\dagger + z^{-2}\mathbf{I} + z^{-4}\mathbf{a}\mathbf{b}^\dagger$ with $\mathbf{a}^\dagger\mathbf{b} = 0$, which is the same as Eq. (6.3) with z replaced by z^2. So it is *cafacafi*. We still have $\mathbf{g}_m(0) = \mathbf{a}\mathbf{b}^\dagger$ and $\mathbf{h}_m(0) = -\mathbf{a}\mathbf{b}^\dagger$ so that degree-one reduction is not possible (as seen in Sec. 6.1). Since $\mathbf{g}_m(1) = \mathbf{0}$, Eq. (6.8) cannot be satisfied. Thus we cannot do degree reduction by two, if we use the building block (6.6). As the degree one building block and the degree two building block we use are the most general *cafacafi* building blocks, the system $\mathbf{G}_4(z)$ cannot be factorized into lower degree *cafacafi* blocks at all.

7. FACTORIZATION OF CAUSAL FIR SYSTEMS HAVING FIR INVERSE

Let $\mathbf{G}_m(z)$ be a causal FIR system, with an FIR inverse (not necessarily anticausal). Then its determinant has the form cz^{-m}, though m does not represent the McMillan degree unless the inverse is anticausal. Unless $\mathbf{G}_m(z)$ is unimodular in z^{-1}, we have $m > 0$, and the determinant is zero for $z = \infty$. In other words, the constant coefficient $\mathbf{g}_m(0) = \mathbf{G}_m(\infty)$ is singular.

Suppose we wish to express $\mathbf{G}_m(z)$ in the form $\mathbf{G}_m(z) = \mathbf{V}_m(z)\mathbf{G}_{m-1}(z)$ where $\mathbf{V}_m(z)$ is the familiar *cafacafi* building block (2.7). Since $\mathbf{G}_{m-1}(z) = \mathbf{V}_m(z^{-1})\mathbf{G}_m(z)$, it is still FIR. From Eq. (2.8) we see that we can force it to be causal by choosing $\mathbf{v}$ such that $\mathbf{v}^\dagger\mathbf{g}_m(0) = 0$. The singularity of $\mathbf{g}_m(0)$ ensures the

† We can of course multiply this with a nonsingular constant matrix, but it can be absorbed in $\mathbf{G}_{m-2}(z)$ in (6.7) and is of no interest.

existence of such nonnull $\mathbf{v}$. The choice of $\mathbf{u}$ is arbitrary except for the requirement $\mathbf{u}^\dagger\mathbf{v} = 1$ in (2.7). For example we can make $\mathbf{u} = \mathbf{v}$ with unit norm in which case $\mathbf{V}_m(z)$ becomes paraunitary.

Since [det $\mathbf{V}_m(z)$] $= z^{-1}$, we have [det $\mathbf{G}_{m-1}(z)$] $= cz^{-(m-1)}$. Thus $\mathbf{G}_{m-1}(z)$ is causal and FIR with the degree of determinant reduced by one. We can repeat this process until we obtain

$$\mathbf{G}_m(z) = \mathbf{V}_m(z)\mathbf{V}_{m-1}(z)\ldots\mathbf{V}_1(z)\mathbf{G}_0(z) \tag{7.1}$$

where $\mathbf{G}_0(z)$ is unimodular (causal and FIR with determinant $c \neq 0$). So we have proved:

Theorem 7.1. Let $\mathbf{G}_m(z)$ be $M \times M$ causal FIR with FIR inverse so that [det $\mathbf{G}_m(z)$] $= cz^{-m}, c \neq 0$. Then we can factorize it as in (7.1) where $\mathbf{V}_m(z) = \left(\mathbf{I} - \mathbf{u}_m\mathbf{v}_m^\dagger + z^{-1}\mathbf{u}_m\mathbf{v}_m^\dagger\right)$, $\mathbf{v}_m^\dagger\mathbf{u}_m = 1$, and $\mathbf{G}_0(z)$ is unimodular in z^{-1}. The matrices $\mathbf{V}_m(z)$ can be chosen to be paraunitary if desired (by taking $\mathbf{u}_m = \mathbf{v}_m$) in which case the product of quantities on the right hand side preceding $\mathbf{G}_0(z)$ is paraunitary. $\diamond$

As in Sec. 4 we can also replace the building blocks $\mathbf{V}_i(z)$ as in (4.6) to obtain a factorization of the form (4.8), where $\mathbf{T}_0$ is replaced with a unimodular remainder $\mathbf{G}_0(z)$. If $\mathbf{V}_i(z)$ are chosen to be paraunitary, then all $\mathbf{T}_i$ in (4.8) will be unitary.

We therefore see that any casual FIR system $\mathbf{G}_N(z)$ with an FIR inverse can be written as $\mathbf{G}_N(z) = \mathbf{G}_{c,a}(z)\mathbf{G}_{c,c}(z)$ where $\mathbf{G}_{c,a}(z)$ is causal FIR with anticausal FIR inverse, and $\mathbf{G}_{c,c}(z)$ is causal FIR with causal FIR inverse. This follows by letting $\mathbf{G}_0(z) = \mathbf{G}_{c,c}(z)$ and lumping the remaining factors on the right side of (7.1) into $\mathbf{G}_{c,a}(z)$. In particular we can let $\mathbf{G}_{c,a}(z)$ be paraunitary without loss of generality. Notice, however, that the degree of $\mathbf{G}_m(z)$ is not, in general, the sum of the degrees of $\mathbf{G}_{c,a}(z)$ and $\mathbf{G}_{c,c}(z)$, so this is not a minimal decomposition.

8. CONCLUDING REMARKS

Many of the previously reported designs for perfect reconstruction filter banks were orthonormal (i.e., the polyphase matrix $\mathbf{E}(z)$ was paraunitary). In the IIR case this meant that if the analysis filters are causal and stable (poles inside the unit circle) then the synthesis filters would be anticausal and stable (poles outside the unit circle). In [1] we argued that for the FIR case, the more general class of biorthonormal systems can be characterized if we can characterize all causal FIR polyphase matrices with anticausal FIR inverse (i.e., all *cafacafi* matrices). More generally, the relevance of systems with anticausal inverses was elaborated in Sec. 1.1 of [1].

The basic similarity between causal systems with anticausal inverses and causal paraunitary systems is fascinating. First, the latter is a special case of the former. Second, the former is characterized by nonsingular realization matrices (for minimal realizations) whereas the latter is characterized by unitary realization matrices (up to similarity). Finally in the FIR case both of these classes have determinant equal to cz^{-N} where N is the McMillan degree. (That is, both of them achieve the maximum value that the degree of a determiant can achieve, viz., the McMillan degree.) In both cases, the most general degree-one FIR building block has the form $\left(\mathbf{I} - \mathbf{u}\mathbf{v}^{\dagger} + z^{-1}\mathbf{u}\mathbf{v}^{\dagger}\right)\mathbf{G}_0$ where $\mathbf{u}^{\dagger}\mathbf{v} = 1$ and $\mathbf{G}_0$ is nonsingular. In the paraunitary case, we further have $\mathbf{u} = \mathbf{v}$ and $\mathbf{G}_0$ is unitary.

The most significant difference between causal systems with anticausal inverses and causal paraunitary systems is that the former cannot in general be factorized into degree one building blocks whereas the latter can be so factorized. This factorization was used in the past [4] for the design and implementation of orthonormal perfect reconstruction filter banks. We saw in Sec. 4 above that a special case of *cafacafi* systems can indeed be factorized into degree-one *cafacafi* building blocks. These are *cafacafi* systems of order one. This factorization gives rise to the biorthonormal lapped transform (*BOLT*) which is a generalization of the lapped orthonormal transform *LOT*.

The *BOLT* is a maximally decimated analysis bank where the polyphase matrix is a first order causal FIR system with anticausal inverse. Since it is a generalization of the lapped orthonormal transform, it provides additional degrees of freedom in the design. It remains to see how to exploit this freedom while designing filter banks for data compression, or for generation of biorthonormal wavelets and so forth. These require further investigation.

There are other problems requiring further investigation. In this paper we introduced two *cafacafi* building blocks, namely the degree one building block (2.7) and the degree two building block (6.6). (These are the most general building blocks we need to consider). We showed that a subclass of *cafacafi* systems, namely the *BOLT* system can be factorized using degree-one building blocks. On the other hand the degree two building block (6.6) cannot be expressed as a product of the degree-one building blocks. Furthermore there exist examples of *cafacafi* systems whose degree cannot be reduced using either of these two building blocks (end of Sec. 6.2). That is, they cannot be expressed as a product of any combination of the two building blocks.

But what does that mean? Perhaps there is a broader class of building blocks which will suffice for factorization; perhaps the number of required building blocks somehow depends on the order and the size $(M \times M)$ of the *cafacafi* matrices. This seems to be an open issue requiring deeper investigation.

In principle the set of all *cafacafi* matrices can be characterized in terms of the realization matrix $\mathcal{R}$ (Eq. (3.1)). For *cafacafi* systems, the matrix $\mathcal{R}$ is invertible, $\mathbf{A}$ has all eigenvalues equal to zero (equivalently $\mathbf{A}^N = \mathbf{0}$ where $\mathbf{A}$ is $N \times N$), and furthermore the matrix $\widehat{\mathbf{A}}$ in the inverse (3.1) has all eigenvalues equal to zero. So $M \times M$ *cafacafi* matrices with degree N are completely characterized by the set of all $(N+M) \times (N+M)$

matrices $\mathcal{R}$ having the following properties: (i) they are nonsingular (ii) the top-left $N \times N$ submatrix $\mathbf{A}$ has all eigenvalues equal to zero, and (iii) the top-left $N \times N$ submatrix $\widehat{\mathbf{A}}$ of $\mathcal{R}^{-1}$ has all eigenvalues equal to zero. Finding a simple analytic way to impose these three restrictions on a constant $(N+M) \times (N+M)$ matrix $\mathcal{R}$ is still an open problem.

Appendix A. Degree of the second order system (6.3)

For arbitrary M, the system (6.3) has degree M. To see this, first consider $\mathbf{U}(z) \triangleq \mathbf{I} + z^{-1}\mathbf{a}\mathbf{b}^\dagger$, with $\mathbf{a}^\dagger\mathbf{b} = 0$. This is unimodular (i.e., [det $\mathbf{U}(z)$] $= c \neq 0$) with $\mathbf{U}^{-1}(z) = \mathbf{I} - \mathbf{a}\mathbf{b}^\dagger z^{-1}$. Clearly the causal FIR system $z^{-1}\mathbf{U}(z)$ has the anticausal FIR inverse $z\mathbf{I} - \mathbf{a}\mathbf{b}^\dagger$, and by construction its determinant is cz^{-M}. In other words, $z^{-1}\mathbf{U}(z)$ is *cafacafi* and its degree is M (Theorem 5.3, [1]). The system (6.3), which is $\mathbf{a}\mathbf{b}^\dagger + z^{-1}\mathbf{U}(z)$ therefore has degree M.

How do we find a structure for $\mathbf{G}(z)$ with only M delays? Since $\mathbf{a}$ and $\mathbf{b}$ are mutually orthogonal vectors, we can define a $M \times M$ unitary matrix of the form $[\mathbf{P} \quad \mathbf{a} \quad \mathbf{b}]$ where $\mathbf{P}$ is $M \times (M-2)$. (For this purpose we assume that $\mathbf{a}$ and $\mathbf{b}$ have unit norm for simplicity.) We then have $\mathbf{I}_M = \mathbf{a}\mathbf{a}^\dagger + \mathbf{b}\mathbf{b}^\dagger + \mathbf{P}\mathbf{P}^\dagger$, so that we can implement $z^{-1}\mathbf{I}_M$ as shown in Fig. A.1(a). If we insert two new branches as shown in Fig. A.1(b), we obtain a realization of (6.3) with M delays. This, therefore, is a minimal realization.

Appendix B. Most general degree-two building block for M=2

We now find the most general 2×2 degree-two *cafacafi* system $\mathbf{G}(z)$ which cannot be factorized into degree-one building blocks. Since $\mathbf{G}(z)$ has degree $= 2$, it has the form

$$\mathbf{G}(z) = \mathbf{g}(0) + z^{-1}\mathbf{g}(1) + z^{-2}\mathbf{g}(2). \tag{B.1}$$

If $\mathbf{g}(2) = \mathbf{0}$, this becomes a *BOLT* and can be factorized (Sec. 4), so we must have $\mathbf{g}(2) \neq \mathbf{0}$. If $\mathbf{g}(0) = \mathbf{0}$, then the degree-reduction condition (2.10) is trivially satisfied because we can first choose $\mathbf{u}$ and then let $\mathbf{v} = \mathbf{u}$. Summarizing, we have $\mathbf{g}(0) \neq \mathbf{0}$ and $\mathbf{g}(2) \neq \mathbf{0}$.

We know that $\mathbf{G}^{-1}(z)$ has degree two in z (end of Sec. 5.1, [1]). So it has the form $\mathbf{H}(z) = \mathbf{h}(0) + z\mathbf{h}(1) + z^2\mathbf{h}(2)$. Since $\mathbf{G}(z)$ cannot be factorized into degree one *cafacafi* systems, we cannot factorize $\mathbf{H}(z^{-1})$ into degree one *cafacafi* systems. So we can modify the argument in the preceding paragraph and obtain $\mathbf{h}(0) \neq \mathbf{0}$ and $\mathbf{h}(2) \neq \mathbf{0}$. If we now equate the like powers of z in $\mathbf{G}(z)\mathbf{H}(z) = \mathbf{I}$ we obtain, among other things, $\mathbf{g}(0)\mathbf{h}(2) = \mathbf{0}$ and $\mathbf{g}(2)\mathbf{h}(0) = \mathbf{0}$. Since none of the matrices is null and all of them are 2×2, this implies that they all have rank one. So we can write

$$\mathbf{G}(z) = \mathbf{u}\mathbf{v}^\dagger + z^{-1}\mathbf{g}(1) + z^{-2}\mathbf{x}\mathbf{y}^\dagger, \tag{B.2}$$

$$\mathbf{G}^{-1}(z) = \mathbf{H}(z) = \mathbf{v}_\perp \mathbf{u}_\perp^\dagger + z\mathbf{h}(1) + z^2 \mathbf{y}_\perp \mathbf{x}_\perp^\dagger, \tag{B.3}$$

for some nonzero vectors $\mathbf{u}, \mathbf{v}, \mathbf{x}, \mathbf{y}, \mathbf{u}_\perp, \mathbf{v}_\perp, \mathbf{x}_\perp$ and $\mathbf{y}_\perp$.

From Lemma 6.1 we know that in the 2×2 case, failure of degree-one reduction implies $\mathbf{h}(0)\mathbf{g}(0) = \mathbf{g}(0)\mathbf{h}(0) = \mathbf{0}$. So we conclude that $\mathbf{v}^\dagger \mathbf{v}_\perp = 0$ and $\mathbf{u}_\perp^\dagger \mathbf{u} = 0$ (hence the notation with subscript $\perp$). Now the condition $\mathbf{G}(z)\mathbf{H}(z) = \mathbf{I}$ implies, in particular, that $\mathbf{v}^\dagger \mathbf{y}_\perp = 0$ and $\mathbf{y}^\dagger \mathbf{v}_\perp = 0$ (since z^2 and z^{-2} terms in the product are zero). Since all vectors are 2×1 and non null, we conclude $\mathbf{y}_\perp = \mathbf{v}_\perp$ and $\mathbf{y} = \mathbf{v}$ up to scale. Similarly from $\mathbf{H}(z)\mathbf{G}(z) = \mathbf{I}$ we conclude that $\mathbf{x} = \mathbf{u}$ and $\mathbf{x}_\perp = \mathbf{u}_\perp$ up to scale. Summarizing, the two matrices *must* have the form

$$\mathbf{G}(z) = \mathbf{u}\mathbf{v}^\dagger + z^{-1}\mathbf{g}(1) + z^{-2} s_1 \mathbf{u}\mathbf{v}^\dagger. \tag{B.4}$$

$$\mathbf{H}(z) = \mathbf{v}_\perp \mathbf{u}_\perp^\dagger + z\mathbf{h}(1) + z^2 s_2 \mathbf{v}_\perp \mathbf{u}_\perp^\dagger, \tag{B.5}$$

for non zero scalars s_1, s_2. Since $\mathbf{v}^\dagger \mathbf{v}_\perp = \mathbf{u}^\dagger \mathbf{u}_\perp = 0$, we see that the condition $\mathbf{G}(z)\mathbf{H}(z) = \mathbf{I}$ implies $\mathbf{g}(1)\mathbf{h}(1) = \mathbf{I}$. So $\mathbf{g}(1)$ and $\mathbf{h}(1)$ are non singular. We can always factor them out, so let us assume $\mathbf{g}(1) = \mathbf{h}(1) = \mathbf{I}$. That is, except for a constant nonsingular factor, we have the form

$$\mathbf{G}(z) = \mathbf{u}\mathbf{v}^\dagger + z^{-1}\mathbf{I} + z^{-2} s_1 \mathbf{u}\mathbf{v}^\dagger. \tag{B.6}$$

$$\mathbf{H}(z) = \mathbf{v}_\perp \mathbf{u}_\perp^\dagger + z\mathbf{I} + z^2 s_2 \mathbf{v}_\perp \mathbf{u}_\perp^\dagger, \tag{B.7}$$

for non zero scalars s_1, s_2. Now by equating the coefficients of z in $\mathbf{G}(z)\mathbf{H}(z) = \mathbf{I}$ we get $\mathbf{u}\mathbf{v}^\dagger = -s_2 \mathbf{v}_\perp \mathbf{u}_\perp^\dagger$. Similarly $\mathbf{v}_\perp \mathbf{u}_\perp^\dagger = -s_1 \mathbf{u}\mathbf{v}^\dagger$. This means, in particular, $\mathbf{u} = \mathbf{v}_\perp$ up to scale, and therefore $\mathbf{u}^\dagger \mathbf{v} = 0$. Summarizing, the most general 2×2 degree-two *cafacafi* system $\mathbf{G}(z)$ which cannot be factorized into degree-one building blocks has the form

$$\mathbf{G}(z) = \left(\mathbf{u}\mathbf{v}^\dagger + z^{-1}\mathbf{I} + z^{-2} s_1 \mathbf{u}\mathbf{v}^\dagger\right)\mathbf{g}(1), \tag{B.8}$$

where $\mathbf{u}^\dagger \mathbf{v} = 0$, $\mathbf{g}(1)$ is arbitrary nonsingular, and s_1 is arbitrary but non zero. Its anticausal FIR inverse is

$$\mathbf{H}(z) = [\mathbf{g}(1)]^{-1}\left(-s_1 \mathbf{u}\mathbf{v}^\dagger + z\mathbf{I} - z^2 \mathbf{u}\mathbf{v}^\dagger\right), \tag{B.9}$$

as we can double check by multiplying $\mathbf{G}(z)$ and $\mathbf{H}(z)$.

Appendix C. Degree two reduction

Given the *cafacafi* system $\mathbf{G}_m(z)$ with inverse $\mathbf{H}_m(z)$ as in (2.5), suppose we wish to perform a degree reduction by two, using the *cafacafi* building block $\mathbf{V}_2(z)$ in (6.6). This means that we wish to find the vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{G}_m(z)$ can be expressed as in (6.7) where $\mathbf{G}_{m-2}(z)$ is *cafacafi*. Since $\mathbf{G}_{m-2}(z) =$

$\mathbf{V}_2^{-1}(z)\mathbf{G}_m(z)$ it is clear that it is already FIR, and so is $\mathbf{G}_{m-2}^{-1}(z) = \mathbf{G}_m^{-1}(z)\mathbf{V}_2(z)$. Only causality of $\mathbf{G}_{m-2}(z)$ and anticausality of $\mathbf{G}_{m-2}^{-1}(z)$ need to be enforced by choice of $\mathbf{u}$ and $\mathbf{v}$. We have

$$\mathbf{G}_{m-2}(z) = \mathbf{V}_2^{-1}(z)\mathbf{G}_m(z) = \Big(-s\mathbf{u}\mathbf{v}^\dagger + z\mathbf{I} - z^2\mathbf{u}\mathbf{v}^\dagger\Big)\Big(\mathbf{g}_m(0) + z^{-1}\mathbf{g}_m(1) + \dots\Big) \tag{C.1}$$

Causality of this requires that the coefficients of z and z^2 be zero, that is, $\mathbf{g}_m(0) - \mathbf{u}\mathbf{v}^\dagger\mathbf{g}_m(1) = \mathbf{0}$, and $\mathbf{u}\mathbf{v}^\dagger\mathbf{g}_m(0) = \mathbf{0}$. Premultiplying the first of the two conditions by $\mathbf{v}^\dagger$ and using $\mathbf{v}^\dagger\mathbf{u} = 0$ we verify that the second requirement is automatically satisfied, so it is sufficient to satisfy $\mathbf{g}_m(0) - \mathbf{u}\mathbf{v}^\dagger\mathbf{g}_m(1) = \mathbf{0}$. This proves the first part in (6.8). Next

$$\mathbf{G}_{m-2}^{-1}(z) = \mathbf{G}_m^{-1}(z)\mathbf{V}_2(z) = \Big(\mathbf{h}_m(0) + z\mathbf{h}_m(1) + \dots\Big)\Big(\mathbf{u}\mathbf{v}^\dagger + z^{-1}\mathbf{I} + sz^{-2}\mathbf{u}\mathbf{v}^\dagger\Big). \tag{C.2}$$

The anticausality of this requires that the coefficients of z^{-1} and z^{-2} be zero. Proceeding as before, we obtain the second part in (6.8).

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List of figures

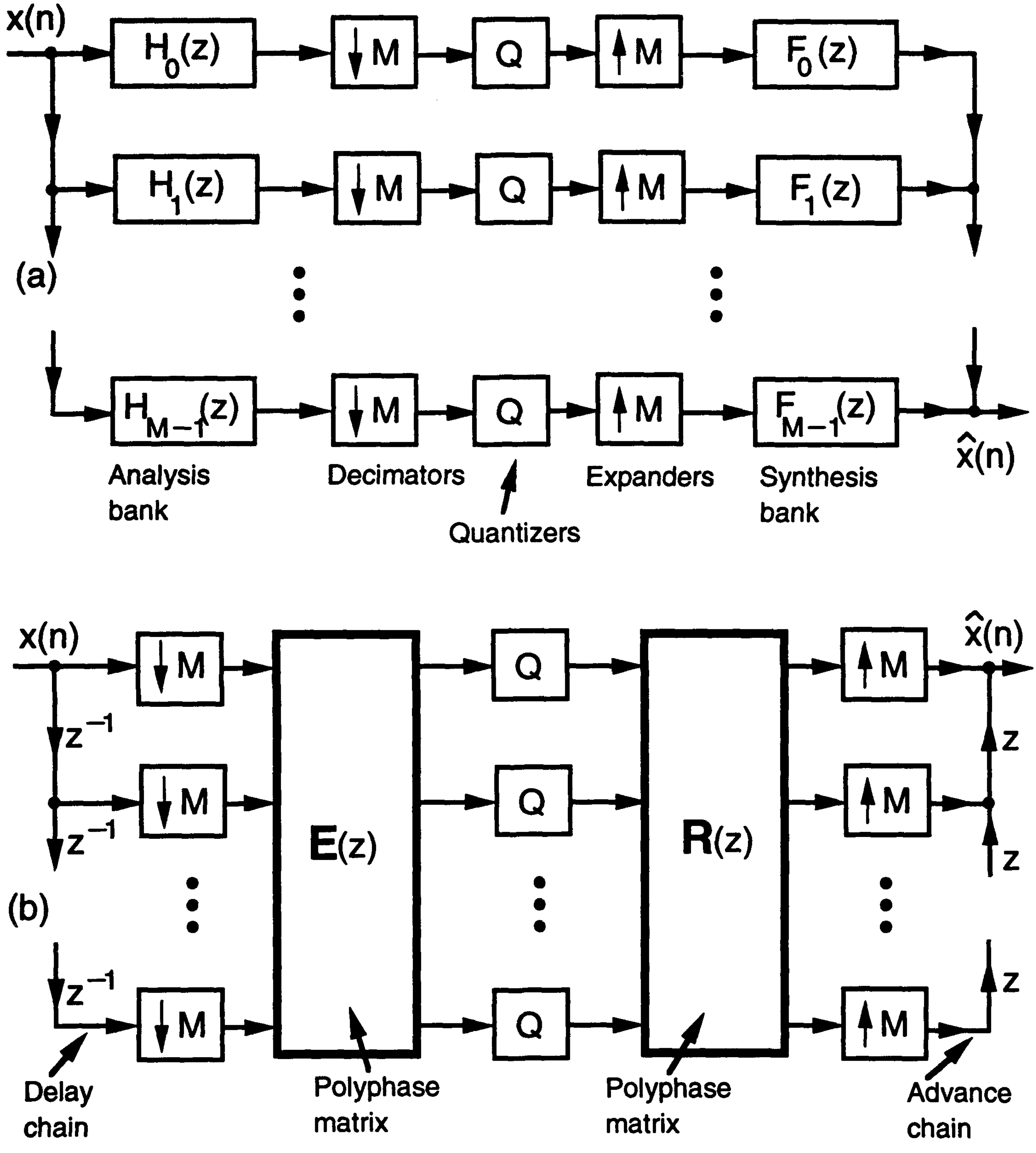

Fig. 1.1 (a) The maximally decimated filter bank, and (b) the polyphase representation.

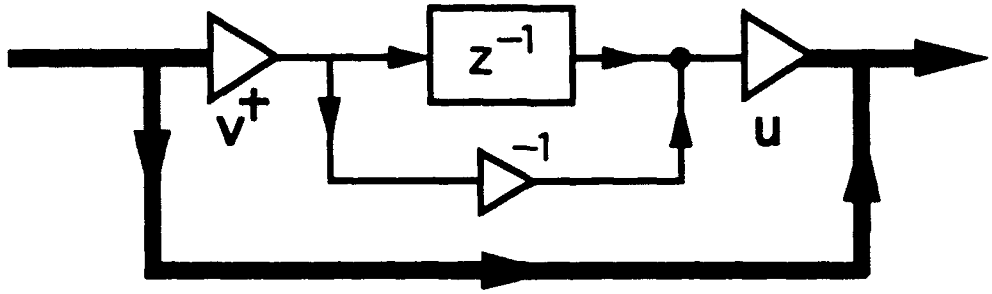

Fig. 2.1. Implementation of the degree-one building block. The thick lines indicate vector signals and thin lines (such as the input and output of the delay element) indicate scalar signals.

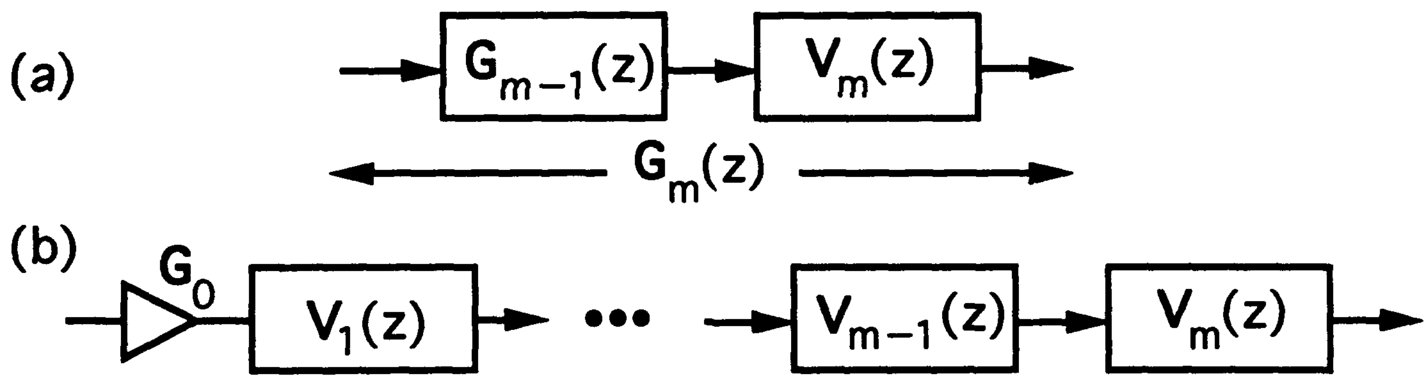

Fig. 2.2 (a) The extraction of a cafacafi building block, and (b) complete cascaded structure.

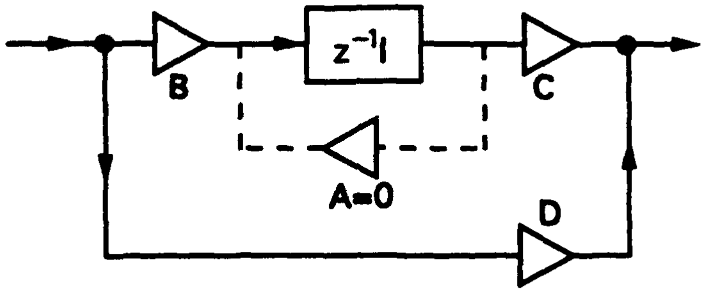

Fig. 4.1. Implementation of $g(0)+z^{-1}g(1)$ where $g(0)=D$ and $g(1)=CB$.

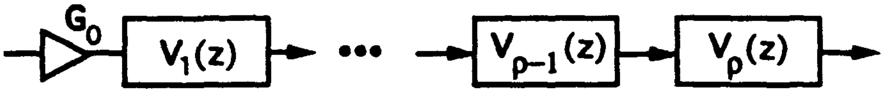

Fig. 4.2. Factorization of the degree-ρ biorthonormal lapped transform (BOLT) in terms of degree-one cafacafi building blocks.

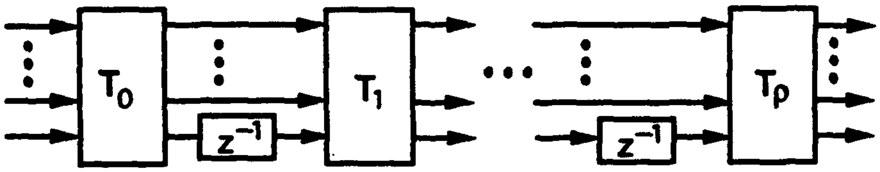

Fig. 4.3. A second factorization of the BOLT in terms of constant nonsingular matrices.

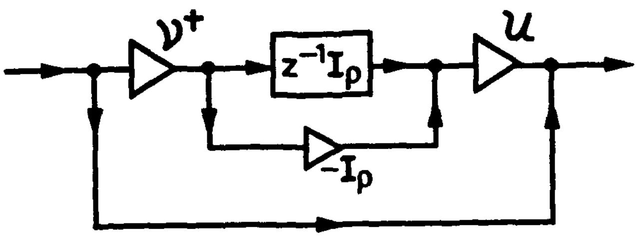

Fig. 5.1. An implementation of the system $F(z)$

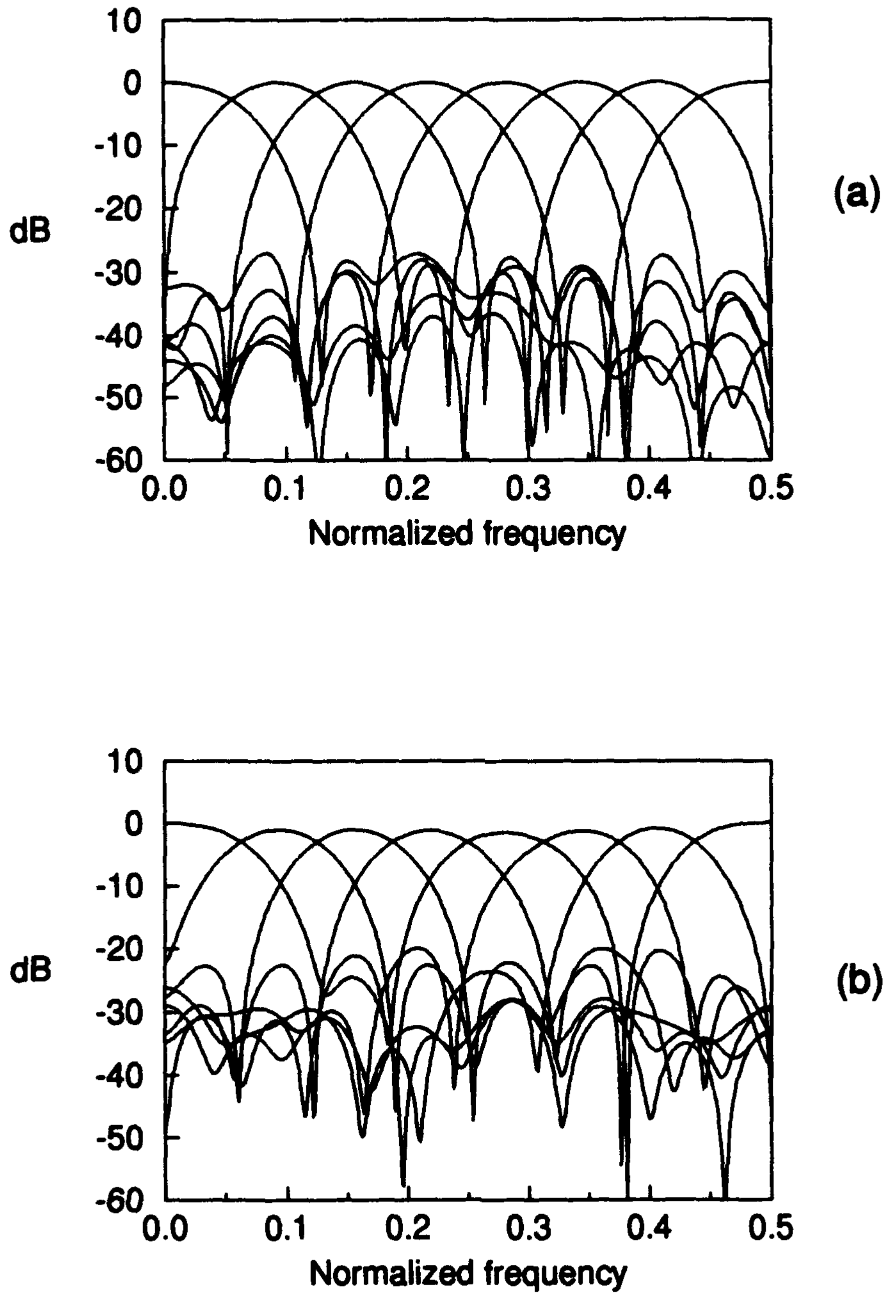

Fig 5.2. Design Example 5.1. Magnitude responses of the analysis filters for an 8-channel filter bank. (a) BOLT filter bank, and (b) LOT filter bank. The degree of the polyphase matrix is 3.

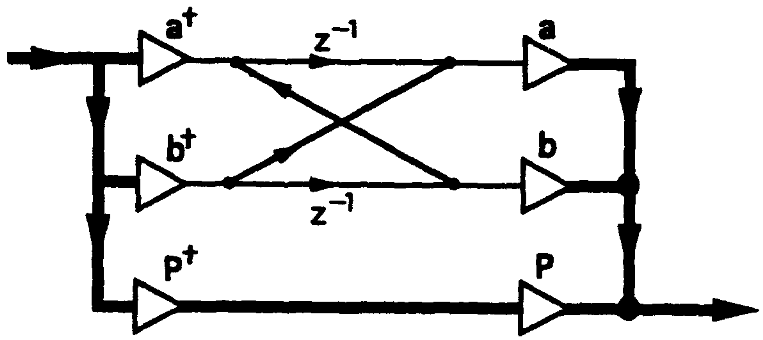

Fig. 6.1. Minimal structure for the MxM degree-2 cafacafi system (6.4).

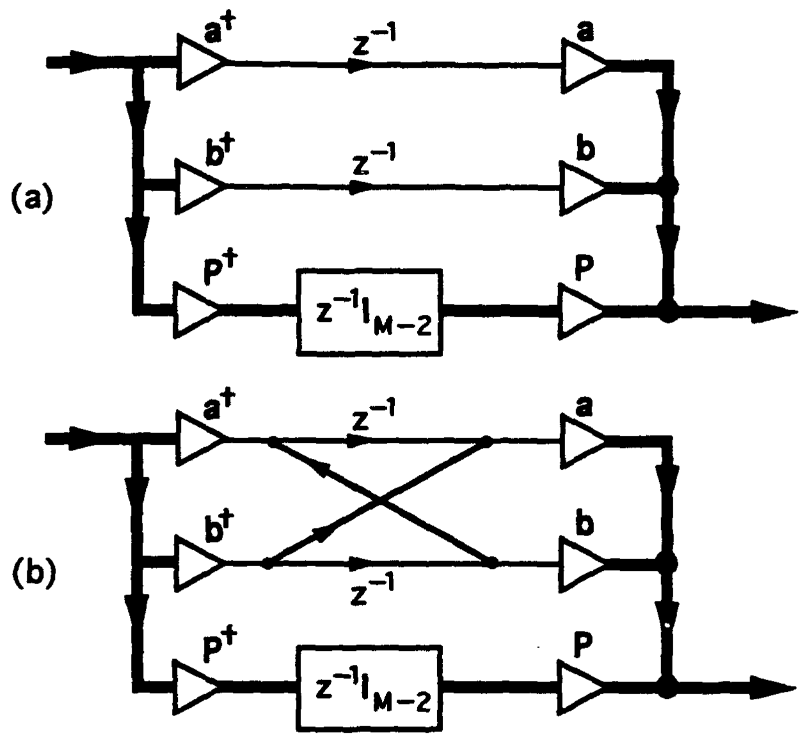

Fig. A.1 (a) Conceptual implementation of $z^{-1}I_M$ and (b) implementation of the degree-M system given by

$$ab^{\dagger} + z^{-1}I_M + z^{-2}ab^{\dagger} \quad (b^{\dagger}a = 0).$$

FOOTNOTES

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